On certain problems of algebraic surfaces

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Sur certains problèmes de surfaces algébriques

Sous la direction de Jinxing Cai et de Qing Liu

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Titre : Sur certains problèmes de surfaces algébriques

Résumé :
La thèse est constituée de deux parties. La première concerne la très amplititude du diviseur canonique relatif, tandis que la seconde traite de la positivité de la caractéristique d'Euler de surfaces.

Dans la première partie, on se donne une courbe régulière propre sur un anneau de Dedekind (dont les corps résiduels aux points fermés sont parfaits), de fibre générique de genre plus grand ou égal à 2. Après contractions de certains diviseurs verticaux, on obtient son modèle canonique. On montre que toute puissance tensorielle supérieure ou égale à 3 du faisceau dualisant relatif sur le modèle canonique est très ample. Ceci améliore un résultat de Jongmin Lee.

Dans la deuxième partie, pour tout nombre premier *p* différent de 2, nous montrons qu'il existe une constante positive *k_p*, telle que pour toute surface projective lisse *X* de type général définie sur un corps algébriquement clos de caractéristique *p,* on ait l'inégalité \xi(*O_X*) ≥ *k_p*c_1^2(*X*).

Mots clés : **Surface, caractéristique d'Euler, diviseur canonique, amplitude.**

Title : On certain problems of algebraic surfaces

Abstract :
This thesis is divided into 2 parts. The first part concerns with the amplitude of relative canonical divisors, and the second part deals with the positivity of the Euler characteristics of surfaces.

In the first part, given a minimal arithmetic surface over a Dedekind ring whose residue fields at closed points are perfect, suppose the general fibre has genus at least 2, after contracting some vertical divisor, we will obtain its canonical model. We prove in this part that 3 or more times the relative canonical divisor of this canonical model is very ample. This simplifies and generalizes a result of Jongmin Lee.

In the second part, we prove that for all prime numbers p>2, there is a positive number k_p, such that \xi(O_X) ≥ k_p c_1^2(X) holds true for all algebraic surfaces X of general type in characteristic p. In particular, \xi(O_X)>0. This answers a question of N. Shepherd-Barron when p>2.

Keywords : **Surface, Euler characteristic, Canonical divisor, ample**

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On Certain Problems of Algebraic Surfaces

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Abstract

This thesis is divided into two parts. The first part concerns with with the amplitude of relative canonical divisors, and the second part deals with positivity of the Euler characteristics of surfaces.

In the first part, given a minimal regular arithmetic surface of fibre genus at least 2 over a Dedekind ring whose residue fields at closed points are perfect, after contracting certain vertical divisors, we obtain its canonical model. In this part, we prove that 3 or more times the relative canonical divisor of the canonical model is very ample. This result simplifies and generalizes a result of Jongmin Lee.

In the second part, we prove that for any prime number $p > 2$, there exists a positive number κ_p such that $\chi(\mathcal{O}_X) \geq \kappa_p c_1^2(X)$ holds true for all algebraic surfaces X of general type in characteristic p. In particular, $\chi(\mathcal{O}_X) > 0$. This answers a question of N. Shepherd-Barron when $p > 2$.

Keywords: surface, canonical divisor, Euler characteristic, ample

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第一章　On pluri-canonical sheaves of arithmetic surfaces

1.1　Introduction

Let S be a Dedekind scheme, let $f : X \to S$ be a minimal regular arithmetic surface of (fibre) genus at least 2 (see [36], §9.3), and let $\omega_{X/S}$ be the relative dualizing invertible sheaf on X. It is well known that $\omega_{X/S}^{\otimes n}$ is globally generated when S is affine and n is big enough (actually $n \geq 2$ suffices, see [32], Theorem 7 and Remark 1.1.3 below). But in general $\omega_{X/S}$ is not relatively ample even when $X \to S$ is semi-stable, because of the possible presence of vertical (-2)-curves ($i.e.$vertical curves C in X such that $\omega_{X/S} \cdot C = 0$). Let $f' : X' \to S$ be the canonical model of X obtained by contracting all vertical (-2)-curves ([36], §9.4.3). Then the relative dualizing sheaf $\omega_{X'/S}$ is a relatively ample invertible sheaf on X' (op. cit., §9.4.20).

Question 1.1.1. *For which $n \in \mathbb{N}$ is $\omega_{X'/S}^{\otimes n}$ relatively very ample ?*

When f is smooth, then $X = X'$ and it is well known that $\omega_{X'/S}^{\otimes n}$ is relatively very ample for all $n \geq 3$. More generally, the same result holds if f is semi-stable (hence $X' \to S$ is stable), see [15], Corollary of Theorem 1.2. In this paper we show that the same bound holds in the general case under the mild assumption that the residue fields of S at closed points are perfect (we do not know whether this condition can be removed). Notice that this condition is satisfied if S is the spectrum of the ring of integers of a number field K, or if S is a regular, integral quasi-projective curve over a perfect field.

Theorem 1.1.2 (Main Theorem). *Let S be a Dedekind scheme with perfect residue fields at closed points, let $f : X \to S$ be a minimal regular arithmetic surface of fibre genus at least 2, and let $f' : X' \to S$ be the canonical model of f. Then $\omega_{X'/S}^{\otimes n}$ is relatively very ample for all $n \geq 3$.*

Remark 1.1.3 Under the above hypothesis and when S is affine, we also give a new proof of a theorem of J. Lee ([32], Theorem 7):

$\omega_{X/S}^{\otimes n}$ (equivalently, $\omega_{X'/S}^{\otimes n}$) is globally generated for all $n \geq 2$.

The proof in [32] is based on a detailed case-by-case analysis. Our method is more synthetic. Moreover, we do not assume that the generic fibre of f is geometrically irreducible, see Proposition 1.3.4. However, as a counterpart, we need the assumption on the perfectness of the residue fields.

The proof of 1.1.2 relies on the following criterion due to F. Catanese, M. Franciosi, K. Hulek and M. Reid.

Theorem 1.1.4 ([12], Theorem 1.1)**.** *Let C be a Cohen-Macaulay curve over an algebraically closed field k, let H be an invertible sheaf on C. Then*

(1) H is globally generated if for every generically Gorenstein subcurve $B \subseteq C$, we have $H \cdot B \geq 2p_a(B)$;

(2) H is very ample if for every generically Gorenstein subcurve $B \subseteq C$, we have $H \cdot B \geq 2p_a(B) + 1$.

Recall that the arithmetic genus is defined by $p_a(B) := 1 - \chi_k(\mathcal{O}_B)$ and that B is called *generically Gorenstein* if its dualizing sheaf is generically invertible. In fact, the first statement of the above theorem is not mentioned in [12], while its proof follows the same routine with that of the second one.

We will apply the above theorem to the closed fibres of f'. The key point is Corollary 1.2.10 which allows to express the arithmetic genus of an effective Weil divisor B on X' as the arithmetic genus of some effective Cartier divisor on X, to overcome the absence of the adjunction formula.

1.2 Arithmetic genus of Weil divisors

In this section we show how to compute the arithmetic genus of an effective Weil divisor on a normal surface with rational singularities. The main result is Corollary 1.2.10.

Let Y be a noetherian normal scheme. We first associate to each Weil divisor $D \in Z^1(Y)$ a coherent sheaf $\mathcal{O}_Y(D) \subseteq \mathcal{K}$, where $\mathcal{K}$ is the constant sheaf $K(Y)$ of rational functions on Y.

Definition 1.2.1 Let $D \in Z^1(Y)$. For any point $\xi \in Y$ of codimension 1, denote by v_ξ the normalised discrete valuation on $K(Y)$ induced by ξ and by $v_\xi(D) \in \mathbb{Z}$ the multiplicity of D at $\overline{\{\xi\}}$. We define the sheaf $\mathcal{O}_Y(D)$ by

$$\mathcal{O}_Y(D)(U) = \{\lambda \in K(Y) \mid v_\xi(\lambda) + v_\xi(D) \geq 0, \quad \forall \xi \in U,\ \operatorname{codim}(\xi, Y) = 1\}.$$

(By convention, $v_\xi(0) = +\infty$).

Remark 1.2.2 (1) Any $\mathcal{O}_Y(D)$ is a reflexive sheaf by [26], Prop.1.6 since it is a normal sheaf.

(2) For any effective Weil divisor $0 < D \in Z^1(Y)$, $\mathcal{O}_Y(-D)$, also denoted by $\mathcal{I}_D$, is a coherent ideal sheaf. In this case, we consider D as *the subscheme of* Y *defined* $\mathcal{I}_D$.

(3) A purely codimension one closed subscheme Z of Y is an effective Weil divisor if and only if it has no embedded points.

Lemma 1.2.3. *Suppose D is an effective Weil divisor on Y. Let Y^0 be an open subset of Y with complement of codimension at least 2, and let Z be any closed subscheme of Y such that $D \cap Y^0 = Z \cap Y^0$. Then $\mathcal{I}_Z \subseteq \mathcal{I}_D$.*

Proof. Let $j : Y^0 \to Y$ be the canonical morphism. We have $j^*\mathcal{I}_Z = j^*\mathcal{I}_D$ by hypothesis. Now the canonical injective morphism $\mathcal{I}_Z \hookrightarrow j_*j^*\mathcal{I}_Z = j_*j^*\mathcal{I}_D$ implies $\mathcal{I}_Z \subseteq \mathcal{I}_D$ because $j_*j^*\mathcal{I}_D = \mathcal{I}_D$ ([26], Prop.1.6). □

From now on we focus on the case of surfaces. More precisely, we suppose $Y = \mathrm{Spec}(R)$ is an affine noetherian normal surface (*i.e.*of dimension 2), with a unique singularity $y \in Y$. Suppose that Y admits a desingularization $g : T \to Y$ (*i.e.*T is regular, g is proper birational and $g : T \setminus g^{-1}(y) \to Y \setminus \{y\}$ is an isomorphism). Let $\{E_i\}_{i=1,\ldots,m}$ be the set of prime divisors contained in $g^{-1}(y)$. We call an effective divisor Z on X *exceptional* if $g(Z) = \{y\}$ as sets, *i.e.*$Z = \sum r_i E_i \geq 0$.

Definition 1.2.4 For any exceptional divisor Z and any coherent sheaf $\mathcal{F}$ on Z, we define the characteristic of $\mathcal{F}$:

$$\chi(\mathcal{F}) := \sum_{i \geq 0} (-1)^i \mathrm{length}_R H^i(Z, \mathcal{F}).$$

We call $p_a(Z) := 1 - \chi(\mathcal{O}_Z)$ the *arithmetic genus of* Z.

This definition makes sense since $H^i(Z, \mathcal{F})$ is a finitely generated R-module supported on y, hence of finite length, and $H^i(Z, \mathcal{F}) = 0$ as soon as $i \geq 2$ since $\dim Z \leq 1$.

Remark 1.2.5 In case T is an open subscheme of an arithmetic surface, and Z is supported in a special fibre, this definition of arithmetic genus coincides with [36], page 431, Equality (4.12).

We recall the definition of rational singularities for surfaces due to M. Artin :

Proposition 1.2.6 ([2], see also [36], §9.4)**.** *The following conditions are equivalent:*

(1) for any exceptional divisor Z, we have $H^1(Z, \mathcal{O}_Z) = 0$;

(2) for any exceptional divisor Z, we have $p_a(Z) \leq 0$;

(3) $R^1 g_*(\mathcal{O}_T) = 0$.

When these conditions hold, y is called a rational singularity*. Note here whether $y \in Y$ is a rational singularity does not depend on the choice of the desingularization $T \to Y$.*

Remark 1.2.7 It is well known that the canonical model X' in Section 1 has only rational singularities ([36], Corollary 9.4.7).

From now on we assume y *is a rational singularity.* We need an important lemma.

Lemma 1.2.8 ([35], Theorem 12.1)**.** *For any $\mathcal{L} \in \mathrm{Pic}(T)$ such that $\deg(\mathcal{L}|_{E_i}) \geq 0$ for all $i \leq m$, $\mathcal{L}$ is globally generated and $H^1(T, \mathcal{L}) = 0$.* □

Proposition 1.2.9. *Let B be an effective Weil divisor on Y, and let $\widetilde{B}$ be its strict transform in T. Then there is an exceptional divisor D on T such that $\mathcal{I}_B\mathcal{O}_T = \mathcal{O}_T(-\widetilde{B} - D)$, $g_*\mathcal{O}_T(-\widetilde{B} - D) = \mathcal{I}_B$ and $R^1g_*\mathcal{O}_T(-\widetilde{B} - D) = 0$.*

Proof. Let $\mathcal{I} := \mathcal{I}_B\mathcal{O}_T$. Let Λ be the set of exceptional divisors Z such that

$$(\widetilde{B} + Z) \cdot E_i \leq 0, \quad \forall i \leq m.$$

Exactly as in [3], Lemma 3.18, one can prove that there is a smallest element D in Λ.

Let $\lambda \in \mathcal{I}_B \setminus \{0\}$. Then $\mathrm{Div}_T(\lambda) = \widetilde{B} + \widetilde{C} + Z$, where Z is exceptional and $\widetilde{C}$ is an effective divisor which does not contain any exceptional divisor. As $\mathrm{Div}_T(\lambda) \cdot E_i = 0$, we have $(\widetilde{B}+Z)\cdot E_i = -E_i \cdot \widetilde{C} \leq 0$. So by definition $Z \in \Lambda$ and hence $\widetilde{B}+Z \geq \widetilde{B}+D$, therefore $\mathcal{I} \subseteq \mathcal{O}_T(-\widetilde{B}-D)$. Since $g_*(\mathcal{O}_T(-\widetilde{B}-D))|_{Y\setminus\{y\}} = \mathcal{I}_B|_{Y\setminus\{y\}}$, it follows from Lemma 1.2.3 that $g_*\mathcal{O}_T(-\widetilde{B} - D) \subseteq \mathcal{I}_B$. Hence

$$\mathcal{I}_B \subseteq g_*\mathcal{I} \subseteq g_*\mathcal{O}_T(-\widetilde{B} - D) \subseteq \mathcal{I}_B.$$

Therefore $g_*\mathcal{O}_T(-\widetilde{B}-D) = \mathcal{I}_B$. By Lemma 1.2.8, $g^*g_*\mathcal{O}_T(-\widetilde{B}-D) \twoheadrightarrow \mathcal{O}_T(-\widetilde{B}-D)$, hence $\mathcal{I} = \mathcal{O}_T(-\widetilde{B} - D)$. The same lemma implies that $R^1g_*\mathcal{O}_T(-\widetilde{B} - D) = H^1(T, \mathcal{O}_T(-\widetilde{B} - D)) = 0$. □

In the next corollary, we use the term "exceptional divisor on X"in the sense of exceptional divisors for $X \to X'$.

Corollary 1.2.10. *(1) Let X, X', S be as in Section 1.1. Let B be an effective vertical Weil divisor on X'. Then there is an exceptional divisor D_B on X such that $p_a(\widetilde{B} + D_B) = p_a(B)$.*

(2) If M' is a normal proper algebraic surface over a field and with at worst rational singularities, and if $g : M \to M'$ is any resolution of singularities, then for any effective Weil divisor B on M', there is an exceptional divisor D_B such that $p_a(\widetilde{B} + D_B) = p_a(B)$.

Proof. (1) Let $g : X \to X'$ be the canonical map. Let $\mathcal{I}_B$ be the ideal sheaf on X' defined as in Remark 1.2.2 (2) and let $\widetilde{B}$ be the strict transform of B in X. By Proposition 2.2.12, there is an exceptional divisor D_B on X such that $g_*\mathcal{O}_X(-\widetilde{B} - D_B) = \mathcal{I}_B$ and $R^1g_*\mathcal{O}_X(-\widetilde{B} - D_B) = 0$. Consider the exact sequence:

$$0 \to \mathcal{O}_X(-\widetilde{B} - D_B) \to \mathcal{O}_X \to \mathcal{O}_{\widetilde{B}+D_B} \to 0.$$

Push-forwarding by g_*, we get:

$$0 \to \mathcal{I}_B \to \mathcal{O}_{X'} \to g_*\mathcal{O}_{\widetilde{B}+D_B} \to R^1g_*\mathcal{O}_X(-\widetilde{B} - D_B) = 0,$$

and

$$0 = R^1g_*\mathcal{O}_X \to R^1g_*\mathcal{O}_{\widetilde{B}+D_B} \to R^2g_*\mathcal{O}_X(-\widetilde{B} - D_B) = 0.$$

So $g_*\mathcal{O}_{\widetilde{B}+D_B} = \mathcal{O}_B$ and $R^1g_*\mathcal{O}_{\widetilde{B}+D_B} = 0$. Hence $\chi(\mathcal{O}_{\widetilde{B}+D_B}) = \chi(\mathcal{O}_B)$, and $p_a(B) = p_a(\widetilde{B} + D_B)$.

(2) The proof is the same as for (1). □

1.3 Proof of the main theorem

Now we can prove the main theorem. We keep the notation of section 1.1.

Lemma 1.3.1. *For any vertical subcurve B_1 of X', there is a vertical effective Weil divisor B on X' and an open subset $U' \subseteq X'$ with complement of codimension at least 2, such that $B \cap U' = B_1 \cap U'$ and $p_a(B) \geq p_a(B_1)$.*

Proof. The curve B_1 has only finitely many embedded points. Let U' be the complement of these points. Now as $B_1 \cap U$ do not admit any embedded points, it is an effective Weil divisor, and extends to a unique effective Weil divisor B on X'.

By Lemma 1.2.3, $\mathcal{I}_{B_1} \subseteq \mathcal{I}_B$ and, since the cokernel is clearly a skyscraper sheaf, we have $\chi(\mathcal{O}_B) - \chi(\mathcal{O}_{B_1}) = -\chi(\mathcal{I}_B/\mathcal{I}_{B_1}) \le 0$, hence $p_a(B) \ge p_a(B_1)$. $\square$

Since in our main theorem we do not assume that the generic fibre is geometrically connected, we need some more preparation. Let

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"'] & S \\ & S' \arrow[u, "\tau"'] \end{tikzcd}$$

be the Stein factorization of f. For any $s \in S$, each connected component of X_s is equal set-theoretically to some fibres of $X \to S'$. The following proposition is well known:

Proposition 1.3.2. *(1) The dualizing sheaf $\omega_{S'/S}$ is invertible, and we have $\omega_{X/S} = g^*\omega_{S'/S} \otimes \omega_{X/S'}$;*

(2) The arithmetic genus of (the generic fiber of) f is strictly larger than 1 if and only if that of g is strictly larger than 1. $\square$

Lemma 1.3.3 ([32], Lemma 4)**.** *Let $s \in S$ be a closed point. Let $\mathcal{L} \in \mathrm{Pic}(X)$. Then $H^1(X_s, \mathcal{L}|_{X_s}) = 0$ if for any effective divisor $0 < A \le X_s$, we have $\mathcal{L} \cdot A > (\omega_{X/S} + A) \cdot A$. In particular, $H^1(X_s, \omega_{X_s/k(s)}^{\otimes n}) = 0$ whenever $n \ge 2$.*

Proof. The first part of the lemma is proved in [32]. We only have to show that $\mathcal{L} := \omega_{X/S}^{\otimes n}$ satisfies the required condition for any $n \ge 2$. If $\omega_{X/S}^{\otimes n} \cdot A \le (\omega_{X/S} + A) \cdot A$, then $\omega_{X/S} \cdot A - A^2 \le 0$ because $\omega_{X/S} \cdot A \ge 0$ by the minimality of $X \to S$, hence $\omega_{X/S} \cdot A = 0, A^2 = 0$, in particular A is the sum of some multiples of fibres of g (see the proof of [36], Theorem 9.1.23). But $\omega_{X/S} \cdot X_{s'} > 0$ for any closed point $s' \in S'$ by Proposition 1.3.2. Contradiction. $\square$

Proof of the main theorem. First we observe that the statement is local on S, so we assume S is local. As the constructions of X and of X' commute with étale base changes, replacing S by its strict henselisation if necessary, we can assume that the residue field k at the closed point $s \in S$ is algebraically closed. Here we use the hypothesis that $k(s)$ is perfect.

By Lemma 1.3.3 and the upper-semicontinuity theorem ([25], III.12), for any $n \geq 2$, we have $R^1 f_*(\omega_{X/S}^{\otimes n}) = 0$. So we obtain $R^1 f'_*(\omega_{X'/S}^{\otimes n}) = 0$ from the exact sequence

$$0 \to R^1 f'_*(\omega_{X'/S}^{\otimes n}) \to R^1 f_*(\omega_{X/S}^{\otimes n})$$

obtained from the Leray spectral sequence for $f_* = f'_* \circ \pi_*$. Again by the upper-semicontinuity theorem we have $H^1(X'_s, \omega_{X'_s/k}^{\otimes n}) = 0$, and the canonical morphism $H^0(X', \omega_{X'/S}^{\otimes n}) \otimes k \to H^0(X'_s, \omega_{X'_s/k}^{\otimes n})$ is an isomorphism.

Let $n \geq 3$. Then it is enough to prove that $\omega_{X'/S}^{\otimes n}|_{X'_s} = \omega_{X'_s/k}^{\otimes n}$ is very ample. Let $C = X'_s$. This is a Cohen-Macaulay curve over k. We will show for any subcurve B_1 of C that

$$n(\omega_{X'_s/k} \cdot B_1) \geq 3(\omega_{X'_s/k} \cdot B_1) \geq 2p_a(B_1) + 1.$$

Theorem 1.1.4 (2) will then imply that $\omega_{X'_s/k}^{\otimes n}$ is very ample. Let B be the effective Weil divisor on X' as given by Lemma 1.3.1. By construction, $p_a(B) \geq p_a(B_1)$. As B and B_1 differ at worst by a zero-dimensional closed subset, we have $\omega_{X'_s/k} \cdot B = \omega_{X'_s/k} \cdot B_1$ (use *e.g.* [19], Prop. 6.2). Therefore it is enough to show the desired inequality for B.

By Corollary 1.2.10, $p_a(B) = p_a(\widetilde{B} + D)$ for some exceptional divisor D. Note in our case that $\omega_{X'_s/k} \cdot B = \omega_{X'/S} \cdot B = \omega_{X/S} \cdot \widetilde{B} = \omega_{X/S} \cdot (\widetilde{B} + D)$ since $\omega_{X/S} \cdot D = 0$ by our assumption. So what we need to prove is:

$$3\omega_{X/S} \cdot (\widetilde{B} + D) \geq 2p_a(\widetilde{B} + D) + 1 = 3 + (\widetilde{B} + D)^2 + \omega_{X/S} \cdot (\widetilde{B} + D).$$

Here we use the adjunction formula on X/S to the divisor $\widetilde{B} + D$. Now it suffices to prove

$$2\omega_{X/S} \cdot (\widetilde{B} + D) - (\widetilde{B} + D)^2 \geq 3.$$

This is true since $\omega_{X/S} \cdot (\widetilde{B} + D) \geq 1$, $-(\widetilde{B} + D)^2 \geq 0$, and if the left-hand side is equal to 2, then $\omega_{X/S} \cdot (\widetilde{B} + D) = 1$ and $(\widetilde{B} + D)^2 = 0$, which is impossible as

$(\widetilde{B}+D)^2+\omega_{X/S}\cdot(\widetilde{B}+D)$ is always even by the adjunction formula on X/S. $\square$

Proposition 1.3.4 ([32], Theorem 7). *Keep the notation of Theorem 1.1.2 and suppose S is affine. Then $\omega_{X/S}^{\otimes n}$ is globally generated for all $n \geq 2$.*

Proof. Using the same argument on upper semicontinuity as the previous proof it suffices to prove that $\omega_{X_s/k}^{\otimes n}$ is globally generated. By Theorem 1.1.4 (1), it is enough to prove that $\omega_{X/S}^{\otimes n}\cdot B \geq 2p_a(B)$ for all $0 < B \leq X_s$. Using the adjuction formula, it is then equivalent to

$$2\omega_{X/S}\cdot B \geq 2+(\omega_{X/S}\cdot B+B^2)$$

which is equivalent to

$$\omega_{X/S}\cdot B - B^2 \geq 2.$$

It is clear that the left-hand side is a non-negative even number, and it is zero only if both $\omega_{X/S}\cdot B=0$ and $B^2=0$. But this is impossible as we pointed out in the proof of Lemma 1.3.3. $\square$

1.4 Another proof of the main theorem in the equal-characteristic case

In this section we give an alternative proof of the main theorem under the assumption that S is of equal-characteristic, *i.e.*, $\mathcal{O}_S$ contains a field. We keep the notation of Section 1. Using Lemma 1.3.3 and the upper semi-continuity theorem ([25], III.12), we find that it is sufficient to prove $\omega_{X'_s/k(s)}^{\otimes n}$ is very ample for $n \geq 3$. Note that this conclusion in fact depends only on X_s: the pluri-canonical morphism restricted to the special fibre is exactly that defined by $H^0(X_s, \omega_{X_s/k}^{\otimes n})$, and $X'_s = \mathrm{Proj}(\bigoplus_i H^0(X_s, \omega_{X_s/k}^{\otimes i}))$ is canonically determined by X_s. In particular, to prove the main theorem we are able to interchange X with another minimal arithmetic surface which has the same special fibre.

The next lemma will allow us to reduce to the case when S is a curve over a field.

Lemma 1.4.1. *Suppose $S = \mathrm{Spec} R$ with $R = k[\![t]\!]$ and k algebraically closed. Let s be the closed point of S. Then there exists a minimal fibration $h : Y \to C$ with Y an integral projective smooth surface of general type over k, C an integral projective smooth curve over k of genus $g \geq 2$, and a closed point $c \in C$ such that X_s is isomorphic to Y_c as k-schemes.*

Proof. Let A be the Henselisation of $k[t]_{tk[t]}$. Then R is the completion of A. In particular $k[t]/t^2 = R/t^2R = A/t^2A$. Let S_1 and V denote the spectra of $k[t]/t^2$ and of A respectively. Thus $\{s\} = \mathrm{Spec} k$ and S_1 are considered as closed subschemes of both S and V.

As $f : X \to S$ is flat and projective, it descends to a flat projective scheme $W \to T$ with T integral of finite type over k, [21], IV.11.2.6. By Artin approximation theorem in [1], there is a morphism $\varphi : V \to T$, such that the diagram below commutes:

$$\begin{tikzcd} S_1 \arrow[r, hook] \arrow[d, hook] & V \arrow[d, "\varphi"] \\ S \arrow[r] & T \end{tikzcd}$$

We claim that $Z := W \times_T V$ is regular. Indeed, $Z_s \simeq X_s$ as s is a closed subscheme of S_1 and of $X \times_S S_1 = W \times_T S_1 = Z \times_V S_1$. With this identification, for any closed point $p \in Z_s$, we have $T_{Z,p} = T_{Z \times_V S_1, p} = T_{X \times_S S_1, p} = T_{X,p}$, where $T_{?,?}$ denotes the Zariski tangent space. As $\dim_k T_{X,p} = 2$ because X is regular, we have $\dim_k T_{Z,p} \leq 2$. By the flatness of Z/V, Z is a 2-dimensional scheme. So Z is regular. Note also that $Z \to V$ is relatively minimal as this property can be checked in the closed fiber and $Z_s \simeq X_s$.

By the construction of V, $Z \to V$ descends to a relatively minimal arithmetic surface $Y_1 \to C_1$ where C_1 is an integral affine smooth curve over k. After compactifying C_1 and Y_1, we find a minimal fibration $Y \to C$ as desired. We take the point $c \in C$ to be the image of $s \in V$ in $C_1 \subset C$. Finally replacing C by some finite cover that is étale at c if necessary, we may assume $g(C) \geq 2$, and consequently Y is of general type. □

Note that when $g(C) \geq 1$, the relative canonical model X/C coincides with the canonical model of X.

Theorem 1.4.2. *Let S, X, X' be as in Section 1. Suppose further that S is an integral projective smooth curve of genus ≥ 2 over an algebraically closed field k. Let $K_{X'}$ be a canonical divisor of X' over k. Then for any sufficiently ample divisor M on S, $nK_{X'} + f'^*M$ is very ample on X' if $n \geq 3$.*

Proof. By Reider's method (see Corollary 1.5.2(3)), we just need to prove that $H^1(X, nK_X + f^*M - 2Z) = 0$, where Z is the vertical fundamental cycle of the (-2)-curves lying above a singularity of X'.

By our assumption on M (being sufficient ample) and applying spectral sequence, it is sufficient to show $R^1 f_* \mathcal{O}_X(nK_X + f^*M - 2Z) = 0$ or, equivalently, that $H^1(X_s, \mathcal{O}_X(nK_X - 2Z)|_{X_s}) = 0$ for any closed point $s \in S$ (note that $\mathcal{O}_X(f^*M)|_{X_s}$ is trivial). As $\mathcal{O}_X(K_X)|_{X_s} \simeq \omega_{X/S}|_{X_s}$, by Lemma 1.3.3 we only need to consider the case when s is the image of Z in S. Again by Lemma 1.3.3, to prove the vanishing above s, it is enough to show that for any divisor $0 < A \leq X_s$,

$$((n-1)K_X - 2Z) \cdot A > A^2.$$

Suppose the contrary. Then

$$2 \geq (A+Z)^2 + 2 \geq (n-1)K_X \cdot A \geq 2K_X \cdot A \geq 0$$

(note that $Z^2 = -2$). This is impossible:

(i) if $K_X \cdot A = 0$, then A consists of (-2)-curves, in particular $Z \cdot A \leq 0$ by the definition of the fundamental cycle, so $(A+Z)^2 \leq A^2 + Z^2 < -2$, contradiction;

(ii) if $K_X \cdot A = 1$, then $(A+Z)^2 = 0$, so $(A+Z) \cdot B = 0$ for any vertical divisor B and thus $A^2 = ((A+Z) - Z)^2 = -2$. This implies that $K_X \cdot A + A^2 = -1$ is odd, contradiction. □

Now we can proceed to the proof of our main results.

Proof of Theorem 1.1.2 and of Proposition 1.3.4. Similarly to the previous section, we can assume $S = \mathrm{Spec} R$ with $R = k[\![t]\!]$ and k algebraically closed. Using Lemma 1.4.1, we can descend $X \to S$ and suppose that S is an integral projective smooth curve over k of genus $g \geq 2$. Let M be a sufficiently ample divisor on S. By Theorem 1.4.2, $nK_{X'} + f'^*M$ is very ample for any $n \geq 3$. Therefore $\omega_{X'_s/k}^{\otimes n} \simeq \mathcal{O}_{X'}(nK_{X'} + f'^*M)|_{X'_s}$ is also very ample. Similarly, if $n \geq 2$, $\omega_{X_s/k}^{\otimes n}$ is globally generated using Corollary 1.5.2 (1). $\square$

Remark 1.4.3 One can also prove Theorem 1.1.2 under the assumptions of 1.4.2 using Theorem 1.2 of [12]. Indeed, replacing C by a finite étale cover of sufficiently high degree if necessary, we may assume $\chi(\mathcal{O}_X) \neq 1$, and $g(C) \gg 0$ so that $p_g(X) \geq 2$ (if $\chi(\mathcal{O}_X) \geq 0$, then $p_g(X) \geq q(X) - 1 \geq g(C) - 1 \gg 2$; if $\chi(\mathcal{O}_X) < 0$, then $p_g(X) \geq -2\chi(\mathcal{O}_X) \geq 2$ by [47], Lemma 13). One then checks that the conditions of [12], Theorem 1.2 are satisfied. Therefore $|nK_{X'}|$ is very ample if $n \geq 3$. In particular $\omega_{X'_s/k}^{\otimes n} \simeq \mathcal{O}_{X'}(nK_{X'})|_{X'_s}$ is also very ample.

1.5 Reider's method

Below is Reider's method ([43], [6], p. 176) in any characteristic.

Theorem 1.5.1 (Reider's Method)**.** *Let X be an integral projective smooth surface over an algebraically closed field, and let L be a nef divisor on X.*

(1) Suppose that any vector bundle E of rank 2 with $\delta(E) := c_1^2(E) - 4c_2(E) \geq L^2 - 4$ is unstable in the sense of Bogomolov ([7]; [6], p. 168). If P is a base point of $|K_X + L|$, then there is an effective divisor D passing through P such that

(a) $D \cdot L = 0$ and $D^2 = -1$; or

(b) $D \cdot L = 1$ and $D^2 = 0$.

(2) Suppose that any vector bundle E of rank 2 with $\delta(E) \geq L^2 - 8$ is unstable in the sense of Bogomolov. If $|K_X + L|$ fails to separate P, Q (possibly infinite near), then there is an effective divisor D passing through P, Q such that

(a) $D \cdot L = 0$ and $D^2 = -2$ or -1; or

(b) $D \cdot L = 1$ *and* $D^2 = 0$ *or* -1*; or*

(c) $D \cdot L = 2$ *and* $D^2 = 0$*; or*

(d) $L^2 = 9$ *and* L *numerically equivalent to* $3D$. □

Corollary 1.5.2. *Keep the above notation and suppose there exists a relatively minimal fibration* $f : X \to S$ *whose fibres have arithmetic genus* ≥ 2 *and such that* $g(S) \geq 2$. *Then:*

(1) $|nK_X + f^*M|$ *is base point free if* $n \geq 2$ *and* M *is sufficiently ample.*

(2) When $n \geq 3$ *and* M *is sufficiently ample,* $|nK_X + f^*M|$ *is very ample outside the locus of vertical* (-2)*-curves and also separates points not connected by vertical* (-2)*-curves.*

(3) Keep the hypothesis of (2). Let $f' : X' \to S$ *be the canonical model of* X *and let* $K_{X'}$ *be a canonical divisor on* X'*. Suppose that for the fundamental cycle* $Z \subset X$ *above any singular point of* X' *we have*

$$H^1(X, \mathcal{O}_X(nK_X + f^*M - 2Z)) = 0,$$

then $nK_{X'} + f'^*M$ *is very ample.*

Proof. (1) - (2) Let $L = (n-1)K_X + f^*M$, so

$$L^2 = (n-1)^2 K_X^2 + 2(n-1)(2g-2)\deg M \gg 0, \quad \text{if } \deg M \gg 0.$$

In characteristic 0, it is well known that any E with $\delta(E) > 0$ is unstable. In positive characteristic p case, we apply [47], Theorem 15, which says that E is semi-stable only if either $pK_X^2 \geq p/2(\sqrt{K_X^2 \cdot \delta}) + 2\chi(\mathcal{O}_X) + p(2p-1)\delta/6$ or $K_X^2 \geq \delta$. So anyway, when M is sufficiently ample, the instability conditions on vector bundles in the above theorem hold. Then by standard discussions we can prove the conditions (a)(b) in 1.5.1(1) will not occur and in 1.5.1(2) only condition (a) where $L \cdot D = 0, D^2 = -2$ can occur, in this case D is a sum of (-2)-curves. So (1), (2) are proved.

(3) We have $H^0(X', \mathcal{O}_{X'}(nK_{X'} + f'^*M)) = H^0(X, \mathcal{O}_X(nK_X + f^*M))$ and the map $X \to \phi_{|nK_X+f^*M|}$ factors through $h : X' \to \phi_{|nK_X+f^*M|}$. Under the assumption

of (2), h is a homeomorphism and is an isomorphism outside the singularities of X'. In order to prove it is an isomorphism it is sufficient to prove that $|nK_{X'} + f'^*M|$ separates tangent spaces of each singularities of X' (see also [11] p. 72). Let x be such a singularity and Z be the fundamental cycle of (-2)-curves lying above x. So what we need is to prove the following sequence is exact:

$$\begin{aligned}0 &\to H^0(X', \mathfrak{m}_x^2\mathcal{O}_{X'}(nK_{X'} + f'^*M)) \to H^0(X', \mathfrak{m}_x\mathcal{O}_{X'}(nK_{X'} + f'^*M)) \\ &\to H^0(X', \mathfrak{m}_x/\mathfrak{m}_x^2 \otimes \mathcal{O}_{X'}(nK_{X'} + f'^*M)) \to 0\end{aligned}$$

where $\mathfrak{m}_x$ denotes the maximal ideal of $\mathcal{O}_{X',x}$. It is enough to show that

$$H^1(X', \mathfrak{m}_x^2\mathcal{O}_{X'}(nK_{X'} + f'^*M)) = 0.$$

Let π denote the canonical morphism from X to X', then it is well known that $\pi_*\mathcal{O}_X(nK_X + f^*M - 2Z) = \mathfrak{m}_x^2\mathcal{O}_{X'}(nK_{X'} + f'^*M)$ ([3], Thm. 3.28). For any irreducible component E of the exceptional locus of $X \to X'$, we have

$$E \cdot (nK_X + f^*M - 2Z) = nE \cdot K_X - 2E \cdot Z = nE \cdot \omega_{X/S} - 2E \cdot Z \geq 0$$

by the minimality of $X \to S$ and because Z is a fundamental cycle. Therefore it follows from Lemma 1.2.8 that $R^1\pi_*\mathcal{O}_X(nK_X + f^*M - 2Z) = 0$, hence

$$H^1(X', \mathfrak{m}_x^2\mathcal{O}_{X'}(nK_{X'} + f'^*M)) = H^1(X, \mathcal{O}_X(nK_X + f^*M - 2Z)) = 0$$

by assumption. □

第二章　On algebraic surfaces of general type with negative c_2

2.1　Introduction

The Enriques-Kodaira classification of algebraic surfaces divides proper smooth algebraic surfaces into four classes according to their Kodaira dimension $-\infty, 0, 1, 2$. A lot of problems remain unsolved for the last class, the so-called *surfaces of general type*. One of the leading problems among these is the following so-called geography problem of minimal surfaces of general type (see [40]).

Question 2.1.1. *Which values of $(a, b) \in \mathbb{Z}^2$ are the Chern invariants (c_1^2, c_2) of a minimal surface of general type ?*

Over the complex numbers, though not yet settled completely, much is known about this problem. Here we collect some classical relations between c_1^2 and c_2 of a minimal surface X of general type:

$$c_1^2 > 0;$$

$$c_1^2 + c_2 \equiv 0 \mod 12;$$

$$5c_1^2 - c_2 + 36 \geq 0; \tag{N}$$

$$3c_2 \geq c_1^2. \tag{BMY}$$

The first inequality is from the definition of a minimal surface of general type, the second condition is from Noether's formula

$$12\chi(\mathcal{O}_X) = c_1^2 + c_2; \tag{2.1.1}$$

the inequality (N) is derived from the following so-called Noether's inequality

$$K_X^2 \geq 2p_g - 4, \tag{2.1.2}$$

here $p_g := h^0(X, K_X)$. The last inequality (BMY) is called the Bogomolov-Miyaoka-Yau inequality. Due to (2.1.1), the inequality (BMY) can also be interpreted as below

(BMY)' $$9\chi(\mathcal{O}_X) \geq c_1^2.$$

It is known that most of the numbers (a, b) satisfying the above relations are the Chern numbers of a surfaces of general type over $\mathbb{C}$. For more details and backgrounds on these inequalities, confer [38], [54], [6] Chap. 7, and [29] Chap. 8 & 9.

Then we turn to the geography problem in positive characteristic cases. Noether's inequality (2.1.2) (see [32]) and Noether's formula (2.1.1) (see [3] Chap. 5) remains true, while Bogomolov-Miyaoka-Yau inequality (BMY) as stated no long holds ([52], § 3.4). In fact, even the following weaker inequality (CdF) due to Castelnuovo and de Franchis fails.

(CdF) $$c_2 \geq 0$$

(see *e.g.* Section 3 of this paper). So it is natural to formulate an inequality in positive characteristics bounding c_2 from below by c_1^2. Using Noether's formula, it is the same as bounding χ from below. In fact, N. Shepherd-Barron has already consider a similar question and proved that $\chi > 0$ (equivalently, $c_2 > -c_1^2$) with a few possible exceptional cases when $p \leq 7$ ([48], Theorem 8). Here we generalize it to the following question.

Question 2.1.2. *What is the optimal number κ_p such that $\chi \geq \kappa_p c_1^2$ holds for all surfaces of general type defined over a field of characteristic p ?*

By definition we have

$$\begin{aligned}\kappa_p = \inf\{\chi/c_1^2 \mid & \text{ minimal algebraic surface of general type defined} \\ & \text{over an algebraically closed field of characteristic } p\}.\end{aligned}$$

In particular, $\kappa_p > 0$ implies $\chi > 0$.

The purpose of this paper is an investigation of κ_p. Of course, it will be in the best situation if we can work out κ_p for each p, however this looks difficult and instead, we try to find some interesting bounds of κ_p, say, to show $\kappa_p > 0$ for all $p > 2$. The main result of this paper is the following theorem.

Theorem 2.1.3 (Main Theorem)**.** *Let κ_p be defined as above, then*

(1) if $p > 2$, $\kappa_p > 0$;

(2) if $p \geq 7$, $\kappa_p > (p-7)/12(p-3)$;

(3) $\lim\limits_{p\to\infty} \kappa_p = 1/12$;

(4) $\kappa_5 = 1/32$.

Moreover, we have a conjecture on the values of κ_p:

Conjecture 2.1.4. *If $p \geq 5$, then $\kappa_p = (p^2-4p-1)/4(3p^2-8p-3)$.*

Note that if $p = 5$, then

$$(p^2-4p-1)/4(3p^2-8p-3) = 1/32,$$

and if $p \geq 7$, then

$$(p^2-4p-1)/4(3p^2-8p-3) > (p-7)/12(p-3).$$

This conjecture comes from the computation of the numerical invariants of Raynaud's examples in [43] (see Subsection 2.3.1). Another computation for a special kind of surfaces of general type is also carried out in the last section of this paper giving some evidence in favour of this conjecture.

In [48], remark after Lemma 9, N. Shepherd-Barron raised the question whether any minimal surface of general type X satisfies $\chi(\mathcal{O}_X) > 0$. Our Theorem 2.1.3 implies that the answer is yes if $p \neq 2$:

Corollary 2.1.5. *If $p \neq 2$, then $\chi > 0$ holds for all surfaces of general type.*

This corollary can help to improve and to better understand several other results (*e.g.* [4], Proposition 2.2, [47], Theorem 25, 26, & 27) where the authors need to take care of the possibility of $\chi \leq 0$.

As another application of Theorem 2.1.3, we give the following theorem concerning the canonical map of surfaces of general type, which can be seen as an analogue of A. Beauville's relevant result over $\mathbb{C}$ ([5], Prop. 4.1, 9.1).

Theorem 2.1.6. *Let S be a proper smooth of surface of general type over an algebraically closed field of characteristic $p > 0$ with $p_g(S) \geq 2$,*

(1) if $p \geq 3$ and $|K_S|$ is composed with a pencil of curves of genus g, then we have

$$g \leq 1 + \frac{p_g + 2}{2\kappa_p(p_g - 1)};$$

(2) if $p \geq 3$ and the canonical map is a generically finite morphism of degree d, then we have

$$d \leq \frac{p_g + 1}{\kappa_p(p_g - 2)}.$$

The proof of this theorem is a naive copy of Beauville's, replacing simply the inequality (BMY)' there by $\chi \geq \kappa_p c_1^2$, hence it will not be included in this paper. The interesting part of this theorem is the following remark.

Remark 2.1.7 If we bound $\chi(\mathcal{O}_S)$ from below (hence it bounds $p_g \geq \chi(\mathcal{O}_S) + 1$ from below) as Beauville did in [5] and substitute κ_p by our lower bounds given in Theorem 2.1.3, we can bound g and d from above as in [5]. As far as I know, whether Beauville's bounds on g and d are optimal is not yet solved, not to mention ours.

We shall also give a new characterization of algebraic surfaces with negative χ due to Theorem 2.1.3.

Theorem 2.1.8 (After [34] Prop. 8.5)**.** *Let S be an algebraic surface over a field of characteristic p with $\chi(\mathcal{O}_S) < 0$. Then,*

(1) S is birationally ruled over a curve of genus $1 - \chi(\mathcal{O}_S)$, or

(2) S is quasi-elliptic of Kodaira-dimension 1 and $p \leq 3$, or

(3) S is of general type and $p = 2$.

We shall briefly explain our idea. Note that once we know that the inequality (CdF) fails in positive characteristics, we immediately obtain $\kappa_p < 1/12$ from (2.1.1) and moreover, in order to study κ_p we only have to consider those surfaces of general type with negative c_2. The main ingredient of this paper is an elaborate study of the numerical invariants of algebraic surfaces of general type with negative c_2 after [48].

This paper is organized as follows.

In Section 2, we give some necessary preliminaries. We rewrite Tate's formula on genus change to obtain some intermediate results which is more or less implicit in both Tate's original paper [53] and [46]. Then we recall the theory of flat double covers, a Bertini type theorem and some other supplements.

In Section 3, we give some examples of algebraic surfaces of general type with negative c_2 and compute some of their numerical invariants.

In Section 4, we study the numerical properties of surfaces of general type with negative c_2, and prove our Theorem 2.1.3 except for the equation $\kappa_5 = 1/32$.

In Section 5, we carry out a calculation of a special kind of algebraic surfaces of general type with negative c_2, namely those X whose Albanese fibration is hyperelliptic and has the smallest possible genus. We show that our conjectural κ_p (Conjecture 2.1.4) are the best bounds of χ/c_1^2 for these surfaces. This also completes the proof of our main theorem. During the calculation, a lemma on a special kind of singularities is used, as the proof is a bit long, we put it as an appendix afterwards this section .

Remark 2.1.9 In this paper we shall use the following notation.

(1) For any invertible sheaf $\mathcal{E}$ over a scheme, $\mathbb{P}(\mathcal{E}) := \mathrm{Proj}(\mathrm{Sym}(\mathcal{E}))$.

(2) If $S \to T$ is a morphism of schemes in characteristic p, we denote by $F_S : S \to S$ the absolute Frobenius morphism and by $F_{S/T} : S \to S^{(p)}$ the relative Frobenius morphism (where $S^{(p)} = S \times_T T$ is obtained by base changing $S \to T$ via $F_T : T \to T$). If $\pi : S \to Y$ is a morphism of T-schemes, we denote by $\pi^{(p)} : S^{(p)} \to Y^{(p)}$ the morphism of T-schemes induced by π.

2.2 Preliminaries

2.2.1 Genus change formula

Let S be a normal projective and geometrically integral curve over a field K (in particular $H^0(S, \mathcal{O}_S) = K$) of positive characteristic p, of arithmetic genus $g(S) := 1 - \chi(\mathcal{O}_S) = \dim H^1(S, \mathcal{O}_S)$. The latter is also called the genus of the function field $K(S)$. Let L/K be a finite extension and let $(S_L)'$ be the normalisation of $S_L := S \times_K L$. A theorem of Tate ([53]) states that

$$(p-1) \mid 2(g((S_L)') - g(S)).$$

This is proved in the scheme-theoretical language in [46]. Below we give a slightly different proof in the scheme-theoretical language (in some places close to Tate's original one) and some more precise intermediate results, in particular, we show that if $g(S)$ is small with respect to p, then the normalisation of $S^{(p)}$ is smooth (Corollary 2.2.8).

Lemma 2.2.1. *Let S, Y be geometrically integral normal curves over a field K of positive characteristic p, let $\pi : S \to Y$ be a finite inseparable morphism of degree p. Then $\Omega_{S/Y}$ is invertible and we have an exact sequence*

$$0 \to F_S^*\Omega_{S/Y} \to \pi^*\Omega_{Y/K} \to \Omega_{S/K} \to \Omega_{S/Y} \to 0 \tag{2.2.1}$$

with $F_S^*\Omega_{S/Y} \simeq \Omega_{S/Y}^{\otimes p}$.

Proof. The second part $\pi^*\Omega_{Y/K} \to \Omega_{S/K} \to \Omega_{S/Y} \to 0$ is canonical and always exact. Let us show the existence of a complex $0 \to F_S^*\Omega_{S/Y} \to \pi^*\Omega_{Y/K} \to \Omega_{S/K}$ and prove the exactness under the assumption of the lemma.

As π is purely inseparable of degree p, we have the inclusions of functions fields $K(S)^p \subseteq K(Y) \subseteq K(S)$, hence $F_{S/K} : S \to S^{(p)}$ factors through $\pi : S \to Y$ and some $f : Y \to S^{(p)}$ (which is in fact the normalisation map). We have a canonical commutative diagram

$$\begin{tikzcd} S \arrow[r, "F_{S/K}"'] \arrow[rr, bend left, "F_S"] \arrow[d, "\pi"'] & S^{(p)} \arrow[r, "q"'] \arrow[d, "\pi^{(p)}"] & S \arrow[d, "\pi"] \\ Y \arrow[r] \arrow[ur, "f"'] & Y^{(p)} \arrow[r] & Y \end{tikzcd}$$

where $q : S^{(p)} \to S$ is the projection map. We have $q^*\Omega_{S/Y} = \Omega_{S^{(p)}/Y^{(p)}}$ because the last square is *Cartesian*, and a canonical map $f^*\Omega_{S^{(p)}/Y^{(p)}} \to \Omega_{Y/Y^{(p)}}$, hence a canonical map $F_S^*\Omega_{S/Y} = \pi^* f^*\Omega_{S^{(p)}/Y^{(p)}} \to \pi^*\Omega_{Y/Y^{(p)}}$. Note that the canonical map $F_{Y/K}^*\Omega_{Y^{(p)}/K} \to \Omega_{Y/K}$ is identically zero, so the canonical map $\Omega_{Y/Y^{(p)}} \to \Omega_{Y/K}$ is an isomorphism. Therefore we have a map $\Phi : F_S^*\Omega_{S/Y} \to \pi^*\Omega_{Y/K}$. Its composition with $\pi^*\Omega_{Y/K} \to \Omega_{S/K}$ is zero because locally it maps a differential form db to $d(b^p) = 0$. So

$$0 \to F_S^*\Omega_{S/Y} \to \pi^*\Omega_{Y/K} \to \Omega_{S/K}$$

is a complex.

Let $s \in S$ and let $y = \pi(s) \in Y$. Then $A := \mathcal{O}_{Y,y} \to B := \mathcal{O}_{S,s}$ is a finite extension of discrete valuation rings of degree p, so $B = A[T]/(T^p - a)$ for some $a \in A$ (the element $a \in A$ is either a uniformizing element or a unit whose class in the residue field of A is a not a p-th power). The stalk of the complex (2.2.1) becomes

$$0 \to Bda \to \Omega_{A/K} \otimes_A B \to ((\Omega_{A/K} \otimes_A B) \oplus BdT)/Bda \to BdT \to 0 \qquad (2.2.2)$$

which is clearly exact. This also shows that $\Omega_{S/Y}$ is locally free of rank 1. As a general fact, we then have $F_S^*\Omega_{S/Y} \simeq \Omega_{S/Y}^{\otimes p}$. $\square$

Proposition 2.2.2. *Let S, Y be normal projective geometrically integral curves over K and let $\pi : S \to Y$ be a finite inseparable morphism of degree p. Let $\mathcal{A} = \mathrm{Ker}(\Omega_{S/K} \to \Omega_{S/Y})$. Then*

(1) $\mathcal{A} = \Omega_{S/K,\mathrm{tor}}$ the torsion part of $\Omega_{S/K}$ and we have

$$(p-1)\deg\det(\mathcal{A}) = 2p(g(S) - g(Y));$$

(2) $\deg\det(\mathcal{A}) = \deg\mathcal{A} = \sum_{s\in S}(\mathrm{length}_{\mathcal{O}_{S,s}}\mathcal{A}_s)[K(s):K] = \dim_K H^0(S,\mathcal{A})$*;*

(3) if $g(S)=g(Y)$*, then* S *is smooth over* K*.*

Proof. (1) Split the exact sequence (2.2.1) into

$$0 \to \Omega_{S/Y}^{\otimes p} \to \pi^*\Omega_{Y/S} \to \mathcal{A} \to 0 \tag{2.2.3}$$

and

$$0 \to \mathcal{A} \to \Omega_{S/K} \to \Omega_{S/Y} \to 0.$$

As $\Omega_{S/K}$ has rank 1 (because S is geometrically reduced) and $\Omega_{S/Y}$ is invertible, we have $\mathcal{A} = \Omega_{S/K,\mathrm{tor}}$. As $\det\Omega_{S/K} = \omega_{S/K}$ and similarly for $\Omega_{Y/K}$, by taking the determinants in the two exact sequences we get

$$\pi^*\omega_{Y/K} = \det\mathcal{A}\otimes\omega_{S/Y}^{\otimes p},$$

and

$$\omega_{S/K} = \det\mathcal{A}\otimes\omega_{S/Y}.$$

Hence

$$(\det\mathcal{A})^{\otimes(p-1)} \simeq \omega_{S/K}^{\otimes p}\otimes\pi^*\omega_{Y/K}^{-1}.$$

By Riemann-Roch, $\deg\omega_{S/K} = 2g(S)-2$ (and similarly for Y). Part (1) is then obtained by taking the degrees in the above isomorphism.

(2) This is well known and can be proved locally at every stalk of $\mathcal{A}$ (see *e.g.*, [37], Lemma 5.3(b)).

(3) Finally, if $g(S)=g(Y)$, then $\deg\mathcal{A}=0$, hence $\mathcal{A}=0$. This implies that $\Omega_{S/K}$ free of rank one, hence S is smooth over K. □

Remark 2.2.3 The support of $\mathcal{A}$ consists of singular (more precisely speaking, non-smooth) points of S, and it is well known that such points are inseparable over K ([36], Proposition 4.3.30). In particular $p \mid [K(s):K]$ for any $s\in\mathrm{Supp}(\mathcal{A})$.

Corollary 2.2.4. (Tate genus change formula) *Let* S *be a normal projective geometrically integral curve over* K*. Let* L *be an algebraic extension of* K *and let* Y *be* *the normalisation of S_L (viewed as a curve over L). Then*

$$p-1 \mid 2(g(S)-g(Y)).$$

Proof. The result will be derived from Proposition 2.2.2. We can suppose L/K is purely inseparable. Let us first treat the case $L = K^{1/p}$. Decompose the absolute Frobenius $K \to K$, $x \mapsto x^p$ as

$$K \xrightarrow{i} K^{1/p} \xrightarrow{\rho} K$$

where i is the canonical inclusion and ρ is an isomorphism. Let us extend Y to $Y_K := Y \otimes_L K$ using ρ. Then Y_K is a normal projective and geometrically integral curve over K, of arithmetic genus (over K) equal to that of Y over L. Moreover Y_K is birational to $(S_L) \otimes_L K = S^{(p)}$. So we have an inseparable finite morphism $S \to Y_K$ of degree p. By Proposition 2.2.2(1), we have $(p-1) \mid 2(g(S)-g(Y))$ and $g(S) > g(Y) \geq 0$ unless S is already smooth over K. Repeating the same argument, for any $n \geq 1$, if S_n denotes the normalisation of $S_{K^{1/p^n}}$, then $p-1$ divides $2(g(S)-g(S_n))$, and S_n is smooth over K^{1/p^n} if n is big enough.

Now let L/K be a finite purely inseparable extension. Then there exists $m \geq 1$ such that $L \subseteq K^{1/p^m} \subseteq L^{1/p^m}$ and S_m is smooth. This implies that the normalisation Y_m of $Y_{L^{1/p^m}}$ is $(S_m)_{L^{1/p^m}}$. On the other hand, applying the previous result to the L-curve Y instead of S, we see that $p-1$ divides $2(g(Y)-g(Y_m))$. As $g(Y_m) = g(S_m)$, we find that $p-1$ divides $2(g(S)-g(Y))$. The case of any algebraic extension follows immediately. □

Lemma 2.2.5. *Let $\pi : S \to Y$ be as in Lemma 2.2.1.*

(1) We have

$$p \deg \Omega_{Y/K,\mathrm{tor}} \leq \deg \Omega_{S/K,\mathrm{tor}}.$$

(2) Let $s \in S$ and let $y = \pi(s)$. Suppose that $K(y) = K(s)$, then

$$\mathrm{length}_{\mathcal{O}_{S,s}} (\Omega_{S/K,\mathrm{tor}})_s \geq p \dim_{K(s)} \Omega_{K(s)/K}.$$

Proof. (1) Denote $\mathcal{A} = \Omega_{S/K,\mathrm{tor}}$ and $\mathcal{B} = \Omega_{Y/K,\mathrm{tor}}$. Let $s \in S$ and $y = \pi(s)$. The canonical map $\mathcal{B}_y \otimes \mathcal{O}_{S,s} = \pi^*(\mathcal{B})_s \to \mathcal{A}_s$ is injective by the exact sequence (2.2.3), because $\Omega_{S/Y}^{\otimes p}$ is torsion-free. Therefore

$$e_s \mathrm{length}_{\mathcal{O}_{Y,y}}(\mathcal{B}_y) = \mathrm{length}_{\mathcal{O}_{S,s}}(\mathcal{B}_y \otimes \mathcal{O}_{S,s}) \leq \mathrm{length}_{\mathcal{O}_{S,s}}(\mathcal{A}_s)$$

where e_s is the ramification index of $\mathcal{O}_{Y,y} \to \mathcal{O}_{S,s}$. The desired inequality holds because $p = e_s[K(s) : K(y)]$.

(2) Let $A = \mathcal{O}_{Y,y}$, $B = \mathcal{O}_{S,s}$. As $K(y) = K(s)$, $A \to B$ has ramification index p. So $B = A[T]/(T^p - t)$ for some uniformizing element t of A. The exact sequence (2.2.2) gives the exact sequence

$$0 \to (\Omega_{A/K}/Adt) \otimes_A B \to \Omega_{B/K} = ((\Omega_{A/K}/Adt) \otimes_A B) \oplus BdT.$$

In particular,

$$\mathcal{A}_s = (\Omega_{A/K}/Adt) \otimes_A B. \tag{2.2.4}$$

The usual exact sequence

$$tA/t^2A \to \Omega_{A/K} \otimes_A K(y) \to \Omega_{K(y)/K} \to 0,$$

implies we have a surjective map

$$\mathcal{A}_s \twoheadrightarrow \Omega_{K(y)/K} \otimes_A B = \Omega_{K(y)/K} \otimes_{K(y)} B/tB.$$

So

$$\mathrm{length}_B \mathcal{A}_s \geq p \dim_{K(y)} \Omega_{K(y)/K} = p \dim_{K(s)} \Omega_{K(s)/K}.$$

□

Corollary 2.2.6. *Let $S = S_0 \to S_1 \to \cdots \to S_n$ be a tower of inseparable covers of degree p of geometrically integral normal projective curves over K. Let $g_i = p_a(S_i)$. Then $g_{i+1} - g_i \leq (g_i - g_{i-1})/p$. In particular* $\deg \Omega_{S/K,\mathrm{tor}} = 2p(g_0 - g_1)/(p-1) > 2(g_0 - g_n)$ *by Proposition 2.2.2.*

Lemma 2.2.5(2) is not used in the sequel. But we think it can be of some interest in the understanding of genus changes. It implies immediately [45], Corollary 3.3.

Definition 2.2.7 We call a curve S over K *geometrically rational* if $S_{\bar{K}}$ is integral with normalisation isomorphic to $\mathbb{P}^1_{\bar{K}}$.

A slightly weaker version of the next corollary can also be found in [45], Corollary 3.2.

Corollary 2.2.8. *Let S be a projective normal and geometrically rational curve over K of (arithmetic) genus g. Suppose that S is not smooth. Let Y be the normalisation of $S^{(p)}$.*

(1) We have $2g \geq (p-1)$. If $2g = p-1$, then Y is a smooth conic, Moreover, S has exactly one non-smooth point, the latter being of degree p over K.

(2) If $g < (p^2-1)/2$, then Y is a smooth conic over K. In particular, we have $\deg \Omega_{S/K,\mathrm{tor}} = 2pg/(p-1)$.

Proof. (1) This is an immediate consequence of Proposition 2.2.2.

(2) If Y is not smooth, as non-smooth points have inseparable residue fields (see *e.g.* [36], Prop 4.3.30), we have $\deg \Omega_{S/K,\mathrm{tor}} \geq p \deg \Omega_{Y/K,\mathrm{tor}} \geq p^2$ by Lemma 2.2.5(1). So $g \geq g(Y)+p(p-1)/2 \geq (p^2-1)/2$ since $g(Y) \geq (p-1)/2$, contradiction. So Y is smooth. In particular, Y is a smooth conic because S is assumed to be geometrically rational. □

2.2.2 Flat double covers

We recall some basic facts on flat double covers. One can also consult [13], §0 or [6], III, §6-7 for a standard introduction.

Definition 2.2.9 A finite morphism between noetherian schemes $f : S \to Y$ is called a *flat double cover* if $f_*\mathcal{O}_S$ is locally free of rank 2 over $\mathcal{O}_Y$.

For our purpose we suppose that Y is an *integral noetherian* scheme defined over a field K of characteristic *different from* 2 in this subsection.

Construction 2.2.10 Flat double covers of Y can be constructed as follows. Choose an invertible sheaf $\mathcal{L}$ on Y, and choose $s \in H^0(Y, \mathcal{L}^{\otimes 2}) = Hom_{\mathcal{O}_Y}(\mathcal{L}^{-2}, \mathcal{O}_Y)$. Endow the $\mathcal{O}_Y$-module $\mathcal{O}_Y \oplus \mathcal{L}^{-1}$ with the $\mathcal{O}_Y$-algebra structure by

$$\mathcal{L}^{-1} \times \mathcal{L}^{-1} \to \mathcal{L}^{\otimes(-2)} \stackrel{e(s)}{\to} \mathcal{O}_Y$$

where $e(s)$ is the evaluation at s. Then $S := \mathrm{Spec}(\mathcal{O}_Y \oplus \mathcal{L}^{-1})$ is a flat double over of Y. Note that if we replace s with $a^2 s$ for some $a \in H^0(Y, \mathcal{O}_Y)^*$, then we get a flat double cover isomorphic to the initial one. We call the invertible sheaf $\mathcal{L}$ above as *the associated invertible sheaf* of f.

Conversely, if $f : S \to Y$ is a flat double cover, we have a trace morphism: $f_*\mathcal{O}_S \to \mathcal{O}_Y$, since $p \neq 2$, this trace morphism splits $f_*\mathcal{O}_S$ into direct sum $\mathcal{O}_Y \oplus \mathcal{L}^{-1}$, where $\mathcal{L}^{-1}$ is the kernel of the trace morphism. Now it is clear that the $\mathcal{O}_Y$-algebra structure of $f_*\mathcal{O}_S$ is given by $\mathcal{L}^{-1} \times \mathcal{L}^{-1} \to \mathcal{O}_Y$ as any elements in $\mathcal{L}^{-1}$ has null trace. For the cover $S \to Y$ defined as above, if $s \neq 0$, S is reduced and $S \to Y$ is generically étale, the *branch divisor* is equal to $B := \mathrm{div}(s)$.

From this construction we immediately obtain the formula of dualizing sheaf.

Corollary 2.2.11. *We have $\omega_{S/Y} = f^*\mathcal{L}$.* □

Corollary 2.2.12. *(1) If $f : Y' \to Y$ is a morphism of integral noetherian schemes, then $S \times_Y Y' \to Y'$ is a flat double cover obtained by $f^*\mathcal{L}$ and $f^*s \in H^0(Y', (f^*\mathcal{L})^{\otimes 2})$.*

(2) If Y is a geometrically connected smooth projective curve over K, and $S \to Y$ is a flat double cover with branch divisor B, then

$$p_a(S) = 2p_a(Y) - 1 + \deg(B)/2.$$

(3) If Y is a geometrically connected smooth projective surface over K, then:

$$\chi(\mathcal{O}_S) = \chi(\mathcal{O}_Y) + \chi(\mathcal{L}^{-1}) = 2\chi(\mathcal{O}_Y) + (B^2 + 2B \cdot K_Y)/8$$

where K_Y is the canonical divisor of Y. □

Proposition 2.2.13. *Let $f : S \to Y$ be a flat double cover over Y with branch divisor B.*

(1) If Y is normal, then S is normal if and only if B is reduced.

(2) If Y is regular, then S is regular if and only if B is regular.

(3) If Y is smooth over K, then S is smooth over K if and only if B is smooth over K.

Proof. See [13], chapter 0. □

Now suppose Y is regular. Let $f : S \to Y$ be a flat double cover given by $\mathcal{L}$ and $s \neq 0$ as in 2.2.10 with $B = \operatorname{div}(s)$ being the branch divisor of f. Then B can be uniquely written as sum of effective divisors: $B = B_1 + 2B_0$ such that B_1 is reduced.

Proposition 2.2.14. *The normalisation of S is the flat double cover $S' \to Y$ given by $\mathcal{L}' := \mathcal{L} \otimes \mathcal{O}_Y(-B_0)$ and $s \in H^0(Y, \mathcal{L}'^{\otimes 2})$ (here we use $\mathcal{L}^{\otimes 2} = s\mathcal{O}_Y(B) \supseteq s\mathcal{O}_Y(B_1) = \mathcal{L}'^{\otimes 2}$, and s is in fact also an global section in $\mathcal{L}'^{\otimes 2}$). Moreover, B_1 is the branch divisor of $S' \to Y$.*

As an application of this proposition, we recall the following process of resolution of singularities from a flat double cover. One may also confer [6], III § 6.

Definition 2.2.15 (Canonical resolution) Let k be an algebraically closed field of characteristic different from 2, and for our purpose, let Y_0 be either a nonsingular algebraic surface over k, or the spectrum of a local ring of a nonsingular algebraic surface over k. Let $f_0 : S_0 \to Y_0$ be a flat double cover given by data $\{\mathcal{L}_0, 0 \neq s \in H^0(Y_0, \mathcal{L}_0^{\otimes 2})\}$ and assume that the branch locus $B := \operatorname{div}(s)$ is reduced (*i.e.* S_0 is normal by Proposition 2.2.13). Then the canonical resolution of singularities of S_0 is the following process:

If B_0 is not regular, choose a singular point $y_0 \in B_0$, let $m_0 := \operatorname{mult}_{y_0} B_0$, and $l_0 := \lfloor m_0/2 \rfloor$. Blowing up y_0 we obtain a morphism $\sigma_0 : Y_1 \to Y_0$. Then $S_0 \times_{Y_0} Y_1 \to Y_1$ is a flat double cover with associated invertible sheaf $\mathcal{L}' = \sigma^* \mathcal{L}_0$ and branch divisor $\sigma_0^* B = \widetilde{B}_0 + m_0 E$, where $\widetilde{B}_0$ is the strict transform of B_0 in Y_1

and E is the exceptional divisor. Let S_1 be the normalisation of $S_0 \times_{Y_0} Y_1$. Then by Proposition 2.2.14, $f_1 : S_1 \to Y_1$ is a flat double cover with associated invertible sheaf $\mathcal{L}_1 = \sigma^*\mathcal{L}_0 \otimes \mathcal{O}_{Y_1}(-l_0 E)$ and branch divisor $B_1 = \sigma^*(B_0) - 2l_0 E$. Replace our data $\{f_0, \mathcal{L}_0, B_0\}$ by $\{f_1, \mathcal{L}_1, B_1\}$ and run the above process again until we reach some n such that B_n is regular, *i.e.* $S_n \to S_0$ is a resolution of singularities by Proposition 2.2.13. To see why this process stops in finitely many times, one may confer [6] Chap. 3.7. We draw the following diagram as a picture of this process. □

$$\begin{tikzcd}
S_0 \arrow[d, "f_0"'] & S_1 \arrow[l, "g_0"] \arrow[d, "f_1"'] & \cdots \arrow[l, "g_1"] & S_n \arrow[l, "g_{n-1}"] \arrow[d, "f_n"] \arrow[lll, bend right, "g"'] \\
Y_0 & Y_1 \arrow[l, "\sigma_0"] & \cdots \arrow[l, "\sigma_1"] & Y_n \arrow[l, "\sigma_{n-1}"]
\end{tikzcd}$$

We will denote by $y_i \in B_i$ the center of the blowing-up morphism $\sigma_i : Y_{i+1} \to Y_i$, E_i the exceptional locus, $m_i := \mathrm{mult}_{y_i} B_i$, and $l_i := \lfloor m_i/2 \rfloor$. Then it follows that

$$\chi(R^1 g_{i*}\mathcal{O}_{S_{i+1}}) = (l_i^2 - l_i)/2. \tag{2.2.5}$$

$$\omega_{S_{i+1}/Y_{i+1}} = g_i^*\omega_{S_i/Y_i} \otimes f_i^*\mathcal{O}_{Y_{i+1}}(-l_i E_i). \tag{2.2.6}$$

In particular, if Y is proper, then

$$\chi(\mathcal{O}_{S_n}) - \chi(\mathcal{O}_{S_0}) = -\sum_{0 \le i < n} (l_i^2 - l_i)/2. \tag{2.2.7}$$

$$K_{S_{i+1}}^2 = K_{S_i}^2 - 2(l_i - 1)^2. \tag{2.2.8}$$

Definition 2.2.16 Given a flat double cover $f_0 : S_0 \to Y_0$ as above, and assume $g : S_n \to S_0$ is the canonical resolution defined as above. Let y be a closed point of the branch divisor B, then there is a unique $s \in S_0$ lying above y, we define $\xi_y := \dim_k R^1 g_*(\mathcal{O}_{S_n})_s$. It is well known that if $g' : \widetilde{S} \to S_0$ is another resolution of singularities, then $\xi_y = \dim_k (R^1 g'_* \mathcal{O}_{\widetilde{S}})_s$.

Keep the notations we introduced for the canonical resolution, then by formula (2.2.5) we can compute ξ_y:

$$\xi_y := \sum_{i \leq n-1} \delta_i(y)(l_i^2 - l_i)/2, \tag{2.2.9}$$

where

$$\delta_i(y) = \begin{cases} 1, & \text{if } y_i \text{ is mapped to } y \text{ by } Y_i \to Y, \\ 0, & \text{otherwise.} \end{cases}$$

By definition, in case Y is projective, we have :

$$\chi(\mathcal{O}_{S_0}) - \chi(\mathcal{O}_{S_n}) = \chi(R^1 g_* \mathcal{O}_{S_n}) = \sum_{y \in B} \xi_y. \tag{2.2.10}$$

Definition 2.2.17 (1) A point $y \in B$ as above is called a negligible singularity of the first kind, if B is locally the union of two nonsingular divisors.

(2) A point $y \in B$ as above is called a negligible singularity of the second kind, if B is locally the union of three nonsingular divisors such that at least two of them meet properly at y.

It is evident from (2.2.9) both kinds of negligible singularities has $\xi_y = 0$. So we are allowed to neglect them in the computation of $\chi(\mathcal{O}_{X_n})$.

Finally we have another application of Proposition 2.2.14.

Definition 2.2.18 In this paper we call a projective curve E over a field K is *hyperelliptic* (resp. *quasi-hyperelliptic*) if it is geometrically integral and admits a flat double cover over $\mathbb{P}^1_K$ (resp. a smooth plane conic).

Proposition 2.2.19. *Let E be a normal projective geometrically rational curve (see Definition 2.2.7) over a field K of characteristic $p \neq 2$. If E is quasi-hyperelliptic, then $p_a(E) = (p^i + p^j - 2)/2$ for some non-negative integer i, j.*

Proof. We can extend K to its separable closure and suppose that K is separably closed. We have a flat double cover $E \to \mathbb{P}^1_K$ with reduced branch divisor B (E is normal). Write $B = b_1 + ... + b_n$. Let $d_i := [k(b_i) : K]$, this is a power of p.

The flat double cover $E_{\bar{K}} \to \mathbb{P}^1_{\bar{K}}$ has its branch divisor $B_{\bar{K}}$ supported in n points, with multiplicities powers of p. By Proposition 2.2.14, the normalisation of $E_{\bar{K}}$ is a flat double cover of $\mathbb{P}^1_{\bar{K}}$ branched at n points. This normalisation being a smooth rational curve, we find $n = 2$ by Corollary 2.2.12(2). So $\deg(B) = d_1 + d_2$ is of the form $p^i + p^j$ and $p_a(E) = \deg(B)/2 - 1$ is of form $(p^i + p^j - 2)/2$ by Corollary 2.2.12(2). □

2.2.3 On a Bertini type theorem

Let S be a proper scheme over a field k, and let $\mathcal{L} = \mathcal{O}_S(D)$ be an invertible sheaf on S. By $|D|$ we denote the set of the effective divisors linearly equivalent to D. Let $H^0(S, \mathcal{O}_S(D))^\vee$ be the dual of the k-vector space $H^0(S, \mathcal{O}_S(D))$. We have a bijection

$$(H^0(S, \mathcal{O}_S(D)) \setminus \{0\})/k^* = \mathbb{P}(H^0(S, \mathcal{O}_S(D))^\vee)(k) \to |D|$$

which maps $s \in H^0(S, \mathcal{O}_S(D)) \setminus \{0\}$ to $D + \mathrm{div}(s)$.

A sub-linear system V of $|D|$ is, by definition, the set of $D + \mathrm{div}(s)$, $s \in \widetilde{V} \setminus \{0\}$, where $\widetilde{V}$ is a linear subspace of $H^0(S, \mathcal{O}_S(D))$, we call this linear system the associated linear system of V. The above bijection establishes a bijection between V and the rational points $\mathbb{P}(\widetilde{V}^\vee)(k)$.

Let $f : X \to C$ be a flat fibration between proper integral varieties over an infinite field k. Let K be the function field of C, and let X_η/K denote the generic fibre of f. Let $\mathcal{L} = \mathcal{O}_X(D)$ be an invertible sheaf on X, and let $V \subseteq |D|$ be a sub-linear system. Denote by D_η the restriction of D to X_η and by V_K the sub-linear system of $|D_\eta|$ generated by the effective divisors D'_η, $D' \in |D|$. The vector space $\widetilde{V}_K$ associated to V_K is exactly $K(i(\widetilde{V})) \subseteq H^0(X_\eta, \mathcal{O}_{X_\eta}(D_\eta))$, where $i : H^0(X, \mathcal{O}_X(D)) \hookrightarrow H^0(X_\eta, \mathcal{O}_{X_\eta}(D_\eta))$ is the canonical restriction map.

Lemma 2.2.20. *Consider the map*

$$r : V = \mathbb{P}(\widetilde{V}^\vee)(k) \to V_K = \mathbb{P}((\widetilde{V}_K)^\vee)(K)$$

defined by $D' \mapsto D'_\eta$. Then r is continuous for the Zariski topology. Moreover, for *any Zariski non-empty open subset U of V_K, $r^{-1}(U)$ is a non-empty Zariski open subset of V.*

Proof. Let $(\widetilde{V}_K)^\vee \hookrightarrow \widetilde{V}^\vee \otimes_k K$ be the dual map of the surjective map $\widetilde{V} \otimes_k K \to \widetilde{V}_K$. It induces a dominant rational map $\mathbb{P}(\widetilde{V}^\vee \otimes_k K) \dashrightarrow \mathbb{P}((\widetilde{V}_K)^\vee)$. Let Ω be the domain of definition of this rational map. Then we see easily that the canonical map $\mathbb{P}(\widetilde{V}^\vee)(k) \to \mathbb{P}(\widetilde{V}^\vee)(K)$ is continuous for the Zariski topology, has image in Ω and the composition $\mathbb{P}(\widetilde{V}^\vee)(k) \to \mathbb{P}(\widetilde{V_K}^\vee)(K)$ is equal to r.

So r is continuous for the Zariski topology. In particular $r^{-1}(U)$ is open. As k and K are infinite, it is well known that $\mathbb{P}(\widetilde{V}^\vee)(k) \hookrightarrow \mathbb{P}(\widetilde{V}^\vee \otimes_k K)(K) = \mathbb{P}(\widetilde{V}^\vee)(K)$ has dense image, and the latter is dense in $\mathbb{P}(\widetilde{V}^\vee \otimes_k K)$. So $r^{-1}(U)$ is non-empty. □

We say that a general member of V has a certain property (P) if there is a non-empty (Zariski) open subset of $\mathbb{P}(\widetilde{V})(k)$ such that each member in this subset satisfies the property (P). This lemma then shows that if a general member of V_K has property (P), so does $D'_\eta := D{=}N_\eta$, where $D' \in V$ is a general member.

Corollary 2.2.21. *Assume $f : X \to C$ is a fibration from a smooth proper surface to a smooth curve over an algebraically closed field, if the generic fibre $X_\eta/K(C)$ is geometrically integral and V is a fix part free linear system on X, let $D \in V$ be a general member, then its horizontal part D_h is reduced and separable over C if the morphism $\phi : X_\eta \to \mathbb{P}(\widetilde{V}_K)$ defined by V_K is separable.*

Proof. Note that D_h is reduced and separable over C if and only if D_η is étale over K. By Lemma 2.2.20, it then suffices to prove that a general member of V_K is étale over K. As V is free of fix part, so is V_K. Therefore a general member of V_K equals to

$$\phi^*(\text{a general hyperplane in } \mathbb{P}(\widetilde{V}_K)).$$

Now since $\phi(X_\eta)$ is geometrically integral (hence only have finitely many non-smooth points over K) and ϕ is separable (hence étale outside finitely many points), a general member of V_K will evidently be étale over K. □

Remark 2.2.22 Let V, D be as above,

(1) if $p \nmid D \cdot F$ (F is a fibre of X/C), then ϕ is automatically separable.

(2) if V is not composed with pencils, then D is furthermore irreducible by [30] Theorem 6.11.

2.2.4 Some other supplementaries

Let k be an algebraically closed field of characteristic p, and $\phi : D \to C$ be *a separable morphism* between two smooth curves over k. Assume $d \in D$ is a closed point and $c := \phi(d)$. Choose an arbitrary uniformizer $s \in \mathcal{O}_{c,C}$ of c.

Definition 2.2.23 We define the *ramification index* of ϕ at d to be the number $R_d(\phi) := \dim_k(\Omega_{D/C})_d$.

And we define the *type of ramification* at d to be a set $\Lambda_d(\phi)$ of numbers as below.

(1) If ϕ is wildly ramified at d, $\Lambda_d(\phi) := \{v(s), R_d(\phi)\}$, where v is the normalised valuation at d. Note here that $v(s)$ is independent on the choice of s and $p \mid v(s)$ by assumption, we also define $j_d(\phi) := v(s)/p$.

(2) If ϕ is tamely ramified at d, $\Lambda_d(\phi) := \{R_d(\phi)\}$. Note that in this case $p \nmid v(s) = R_d(\phi) + 1$. □

When no confusion can occur, we shall use R_d and Λ_d instead of $R_d(\phi)$ and $\Lambda_d(\phi)$.

Remark 2.2.24 By abuse of language we can also talk about the ramification index and ramification type of a certain kind of function as below. Suppose $s \in \mathcal{O}_{d,D} \backslash \mathcal{O}_{d,D}^p$ is an element in the maximal ideal of $\mathcal{O}_{d,D}$, then we can define a separable local morphism (still denoted by s) $s : \mathrm{Spec}(\mathcal{O}_{d,D}) \to \mathrm{Spec}(k[x])_{(x)}$ mapping $x \mapsto s$. By mixing the function s and the associated morphism s we are allowed to talk about its ramification index $R_d(s)$ and ramification type $\Lambda_d(s)$.

From our definition of ramification index, we have Hurwitz's formula:

Proposition 2.2.25 (see, [36] Theorem 4.16 and Remark 4.17)**.** *Suppose $\phi : D \to E$ is a separable morphism between smooth projective curves. Then*

$$2\deg\phi(g(E)-1)+\sum_d R_d(\phi)=2g(D)-2. \tag{2.2.11}$$

□

Finally, to close this section, we shall recall the following variation of Clifford's theorem.

Lemma 2.2.26 ([5], Lemme 1.3)**.** *Let C be a smooth projective curve of genus $q := g(C)$, and let $D \geq 0$ be an effective divisor, then either*

(1) $\deg D > 2(q-1)$*, and* $\deg D = h^0(\mathcal{O}_C(D)) + q - 1$*; or*

(2) $2(h^0(\mathcal{O}_C(D)) - 1) \leq \deg D \leq 2(q-1)$*. In particular this time we have* $h^0(\mathcal{O}_C(D)) \leq q$*.*

2.3 Examples

In this section we will present some examples of surfaces of general type with negative c_2 and calculate some of their numerical invariants.

2.3.1 Examples of M. Raynaud

Let us briefly recall the examples of M. Raynaud [43].

Let k be an algebraically closed field of characteristic $p > 2$, and assume C is a smooth projective curve of genus $q \geq 2$ such that there is an $f \in K(C)$ satisfying $(df) = pD$ for some divisor D. Let $\mathcal{L} = \mathcal{O}_C(D)$, $l = \deg D$ and $\mathcal{M}$ be any invertible sheaf on C such that $\mathcal{M}^{\otimes 2} \simeq \mathcal{L}$. We have $m := \deg\mathcal{M} = l/2$ and $2q-2 = pl = 2pm$.

By [43] Proposition 1, we can find a rank 2 locally free sheaf $\mathcal{E}$ and its associated ruled surface $\rho : Z := \mathbb{P}(\mathcal{E}) \to C$ such that

(1) $\det(\mathcal{E}) \simeq \mathcal{L}$, in particular $\mathcal{O}(1)^2 = l$;

(2) there is a section $\Sigma_1 \in |\mathcal{O}(1)|$;

(3) there is a multi-section Σ_2 such that the canonical morphism $\rho : \Sigma_2 \to C$ is isomorphic to the Frobenius morphism.

(4) $\Sigma_1 \cap \Sigma_2 = \emptyset$,

(5) $\mathcal{O}_Z(\Sigma_2) = \mathcal{O}(p) \otimes \rho^*(\mathcal{L}^{\otimes -p})$.

Let $\Sigma := \Sigma_1 + \Sigma_2$, then Σ is a nonsingular divisor of Z, and

$$\mathcal{O}_Z(\Sigma) = \mathcal{O}(p+1) \otimes \rho^*(\mathcal{L}^{\otimes -p}) = (\mathcal{O}(\frac{p+1}{2}) \otimes \rho^*(\mathcal{M}^{\otimes -p}))^{\otimes 2},$$

hence the data $\{\mathcal{O}(\frac{p+1}{2}) \otimes \rho^*(\mathcal{M}^{\otimes -p}), \Sigma \in |(\mathcal{O}(\frac{p+1}{2}) \otimes \rho^*(\mathcal{M}^{\otimes -p}))^{\otimes 2}|\}$ defines a flat double cover $\pi : S \to Z$ by Construction 2.2.10.

Proposition 2.3.1. *We have*

(1) $K_Z = \mathcal{O}(-2) \times (\rho^* \mathcal{L}^{\otimes p+1})$*, and* $K_S = \pi^*(\mathcal{O}(\frac{p-3}{2}) \otimes \rho^* \mathcal{M}^{\otimes p+2})$*;*

(2) $\chi(\mathcal{O}_S) = (p^2 - 4p - 1)l/8$*,* $K_S^2 = (3p^2 - 8p - 3)l/2$*, and* $c_2(S) = -4(q-1)$*;*

(3) S *is a minimal surface of general type if* $p \geq 5$*.*

Proof. By Proposition 2.2.13, S is regular.

(1) Since $\det \mathcal{E} = \mathcal{L}$, $\Omega_{C/k} \simeq \mathcal{L}^{\otimes p}$, we immediately get

$$K_Z = \mathcal{O}(-2) \otimes \rho^* \mathcal{L}^{\otimes p+1},$$

then by Corollary 2.2.11,

$$\omega_{S/Z} = \pi^*(\mathcal{O}(\frac{p+1}{2}) \otimes \rho^* \mathcal{M}^{\otimes -p}),$$

hence

$$K_S = \pi^*(\mathcal{O}(\frac{p-3}{2}) \otimes \rho^* \mathcal{M}^{\otimes p+2}).$$

(2) By Corollary 2.2.12, we have

$$\chi(\mathcal{O}_S) = 2\chi(\mathcal{O}_Z) + \frac{\Sigma^2 + 2\Sigma \cdot K_Z}{8} = \frac{p^2 - 4p - 1}{8} l,$$

and

$$K_S^2 = \pi^*(\mathcal{O}(\frac{p-3}{2}) \otimes \rho^*\mathcal{M}^{\otimes p+2})^2 = 2(\mathcal{O}(\frac{p-3}{2}) \otimes \rho^*\mathcal{M}^{\otimes p+2})^2 = \frac{3p^2-8p-3}{2}l,$$

therefore $c_2(S) = 12\chi(\mathcal{O}_S) - K_S^2 = -2pl = -4(q-1)$.

(3) If $p \geq 5$, then any closed fibre of $S \to C$ is irreducible and has arithmetic genus $(p-1)/2$, hence S is a minimal surface of general type. □

Remark 2.3.2 (1) Note that the fibration $S \to C$ is uniruled. In this case we do not have the positivity of the dualizing sheaf $\omega_{S/C}$ (compare with [51] § 2). We shall point out that $\omega_{S/C}$ here is *not nef*. In fact $\omega_{S/C} = \omega_{S/Z} \otimes \pi^*\omega_{Z/C} = \pi^*(\mathcal{O}(\frac{p-3}{2}) \otimes \rho^*\mathcal{M}^{\otimes 2-p})$, however

$$\Sigma_1 \cdot (\mathcal{O}(\frac{p-3}{2}) \otimes \rho^*\mathcal{M}^{\otimes 2-p}) = -l/2 < 0.$$

(2) Note that

$$\frac{\chi(\mathcal{O}_X)}{K_X^2} = \frac{p^2-4p-1}{4(3p^2-8p-3)}.$$

This number is exactly our conjectural κ_p (Conjecture 2.1.4).

(3) If $p = 3$, Raynaud's example is an quasi-elliptic surface, hence it is not of general type. This is one of the reasons why we can find κ_5 but not κ_3.

2.3.2 Examples in characteristic $2, 3$

First we give an example of surfaces with negative c_2 over a field k of characteristic 3. Choose $m = 3^n - 1$ points, say $t_1, ..., t_m$ on $\mathbb{A}_k^1 = \mathbb{P}_k^1 \backslash \{\infty\}$, and we can construct a cyclic cover $C \to \mathbb{P}_k^1$ of degree m such that the branch locus equals to $B := \sum_i t_i$ canonically as we did before for flat double covers (see Construction 2.2.10). In particular by Hurwitz's formula, $q - 1 := g(C) - 1 = (3^n-1)(3^n-4)/2$

Let $Y := \mathbb{P}_C^1$, $p_1 : Y \to C$, and $p_2 : Y \to \mathbb{P}_k^1$ be the canonical projections. Let Π_1 be the divisor $C \times_k \{\infty\}$, and Π_2 be the divisor which is the image of $C \xrightarrow{F^n \times h} C \times_k \mathbb{P}_k^1 = Y$, here F^n is the n-th Frobenius morphism. Then $\Pi := \Pi_1 + \Pi_2$

is an even divisor (*i.e.*, $\Pi = 2D$ for some divisor D), in particular we can define a flat double cover $\pi : S' \to Y$ whose branch locus equals to Π.

Proposition 2.3.3. *Let S be the minimal model of S', when $n \geq 2$, S is of general type and $c_2(S) \leq -4(q-1) + 3m$.*

Sketch of the proof. We consider the canonical resolution of S. We have Π_1 and Π_2 intersect properly, and the singularities of Π_2 are the pre-images of B. Blowing up these points($2m$ points in total), we get the desingularization of Π. Consequently we get a desingularization $S_1 \to S'$. It is clearly $S_1 \to C$ has $2m$ non-irreducible fibres (each has 2 components), therefore we have

$$c_2(S) \leq c_2(S_1) = -4(q-1) + 3m$$

by Grothendieck-Ogg-Shafarevich formula (see formula (2.4.2) below). □

Remark 2.3.4 When $n \to +\infty$, we see that $c_2(S)/(q-1) \to -4$.

We mention that in characteristic 2 there are also surfaces of general type with negative c_2. One example is [33], Theorem 7.1, where $c_1^2 = 14, \chi = 1$ and $c_2 = -2$.

2.4 Surfaces of general type with negative c_2

Let k be any algebraically closed field of characteristic $p > 0$, and let X be a minimal surface of general type with negative $c_2(X)$. We first recall a theorem of N. Shepherd-Barron on the structure of the Albanese morphism of X.

Theorem 2.4.1 ([48] Theorem 6)**.** *The Albanese morphism of X factors through a fibration $f : X \to C$ such that:*

(1) C is a nonsingular projective curve of genus $q := g(C) \geq 2$, $f_*\mathcal{O}_X \simeq \mathcal{O}_C$, and $\mathrm{Alb}_X \simeq \mathrm{Alb}_C$.

(2) The geometric generic fibre of f is an integral singular rational curve with unibranch singularities only.

We then introduce the following notation according to this theorem:

(1) $K := K(C)$ (resp. $\overline{K}$; η; $\overline{\eta}$) is the function field of C (resp. a fixed algebraic closure of K; the generic point of C; a fixed geometric generic point of C);
(2) F : a general fibre of f;
(3) $g := p_a(F)$ is the arithmetic genus of any fibre of f;
(4) $p_g := h^2(X, \mathcal{O}_X)$ is the geometric genus of X.
(5) $q(X) := h^1(X, \mathcal{O}_X)$ is the irregularity of X. Since $\mathrm{Alb}_X \simeq \mathrm{Alb}_C$, we have the following inequality due to Igusa [27],

$$q(X) \geq \dim \mathrm{Alb}_X = \dim \mathrm{Alb}_C = q; \tag{2.4.1}$$

(6) Denote by Z the fixed part of $|K_X|$, Z_h the horizontal part of Z and $Z_0 := (Z_h)_{\mathrm{red}}$;
(7) Let $f^*(\Omega_{C/k})(\Delta)$ be the saturation of the injection $f^*\Omega_{C/k} \to \Omega_{X/k}$. Define $N := f^*K_C + \Delta$ to be the divisor class of $f^*(\Omega_{C/k})(\Delta)$. It is well known that Δ is supported on the non-smooth locus of f, in particular each prime horizontal component of Δ is inseparable over C.
(8) For any effective divisor D on X, we will use both D_η and $D|_{X_\eta}$ to denote its restriction to the generic fibre of f and we use D_h, D_v to denote its horizontal and vertical part.

If let $S := X_\eta/K$, then by our construction we have $\mathcal{O}_{\Delta_\eta} \simeq \mathcal{A} := (\Omega_{X_\eta/K})_{\mathrm{tor}}$ and $\mathcal{O}_X(\Delta)|_{X_\eta} \simeq \det \mathcal{A}$. Therefore Corollary 2.2.8 implies the following lemma.

Lemma 2.4.2. *We have*

(1) $(p-1) \mid 2g$;
(2) if $g < (p^2-1)/2$, *then* $\deg_K(\Delta_\eta) = 2pg/(p-1)$; *In particular, if* $g = (p-1)/2$, *then* Δ_h *is integral.*

From Noether's formula (2.1.1), to bound κ_p from below, we only have to bound $\lambda(X) := K_X^2/(q-1)$ and $\gamma(X) := c_2(X)/(q-1)$. One lower bound of $\gamma(X)$ comes out naturally once we apply Grothendieck-Ogg-Shafarevich formula ([22], Exposé

X) to X to obtain the following formula:

$$c_2(X) = -4(q-1) + \sum_{c\in|C|} (b_2(X_c) - 1) \geq -4(q-1). \tag{2.4.2}$$

Here we note that $H^1_{\text{ét}}(X_{\overline{\eta}}, \mathbb{Q}_l) = 0$, as $X_{\overline{\eta}}$ is a rational curve with unibranch singularities only, hence the Swan conductor and $b_1(X_c)$ both vanish. By the way, this formula also shows that X is supersingular in the sense of Shioda.

Proposition 2.4.3. *The surface X is supersingular in the sense that $b_2(X) = \varrho(X)$, here $\varrho(X)$ is the Picard number of X.*

Proof. Using the fibration $f : X \to C$ we have

$$\varrho(X) \geq 2 + \sum_{c\in|C|} ({}^\sharp\{\text{irreducible components of } X_c\} - 1) = 2 + \sum_{c\in|C|} (b_2(X_c) - 1).$$

Conversely since $b_1(X) = 2q$ and $c_2(X) = 2 - 2b_1 + b_2$, we get

$$b_2 = 2 + \sum_{c\in|C|} (b_2(X_c) - 1) \leq \varrho(X)$$

from the (2.4.2). Hence $b_2 = \varrho(X)$ and X is supersingular. □

Remark 2.4.4 Since X is dominated by a ruled surface, Proposition 2.4.3 can also be derived from Lemma of [50] §2.

It remains to find lower bounds of $\lambda(X) = K_X^2/(q-1)$. Note that pulling back by an étale cover of C, $\lambda(X)$ is invariant while $q-1$ and K_X^2 are multiplied by the degree of the cover, thus we can assume

$$q \gg \lambda(X) > 0,\ K_X^2 \gg 0.$$

We shall first go through N. Shepherd-Barron's method in [48] quickly, based on which we will give an improvement.

Lemma 2.4.5. *Assume H is a reduced horizontal divisor on X such that any of its irreducible component is separable over C, then we have:*

$$N \cdot H \leq (H + K_X) \cdot H.$$

Proof. We consider the morphism $\mathcal{O}_X(N)|_H \to \omega_{H/k}$ given by the composition $\mathcal{O}_X(N)|_H = f^*(\Omega_{C/k})(\Delta)|_H \to \Omega_{X/k}|_H \to \Omega_{H/k} \to \omega_{H/k}$. We show that under our assumption this morphism is injective. Taking the degrees in $\mathcal{O}_X(N)|_H \hookrightarrow \omega_{H/k}$ will then imply that $N \cdot H \leq \deg(\omega_{H/k}) = (K_X + H) \cdot H$.

Let $\xi_i \in X_\eta$ be the generic point of an irreducible component H_i of H. Then ξ_i belongs to the smooth locus of X_η/K, so $(\mathcal{O}_X(N)|_H)_{\xi_i} \to \omega_{H/k,\xi_i}$ coincides with $(f^*\Omega_{C/k})_{\xi_i} \to \Omega_{H/k,\xi_i}$ and the latter is injective because $H_i \to C$ is separable. So the kernel of $\mathcal{O}_X(N)|_H \to \omega_{H/k}$ is a skyscraper sheaf. As $\mathcal{O}_X(N)$ is an invertible sheaf and H has no embedded points (it is locally complete intersection), the kernel is trivial and $\mathcal{O}_X(N)|_H \to \omega_{H/k}$ is injective. □

Corollary 2.4.6. *If the complete linear system $|H|$ is free of fixed part and defines a separable generically finite map, then $N \cdot H \leq (H + K_X) \cdot H$.*

Proof. It suffices to show that a general member of $|H|$ is integral and separable over C, but this follows immediately from Corollary 2.2.21. □

With the help of [47] Theorems 24, 25, 27 and under our assumption $q \gg \lambda(X) > 0, K_X^2 \gg 0$, we then see that the linear systems

(1) $|2K_X|$, for $p > 2, g > 2$;

(2) $|3K_X|$, for $p = 2, g > 2$;

are base point free and define birational morphisms. Applying Lemma 2.4.5 to the above linear systems, we then obtain:

Corollary 2.4.7. *(1) If $p \geq 3, g \geq 3$, then*

$$4(g-1)(q-1) + K_X \cdot \Delta_h \leq 3K_X^2. \tag{2.4.3}$$

(2) If $p=2, g\geq 3$, then

$$4(g-1)(q-1)+K_X\cdot\Delta_h\leq 4K_X^2. \tag{2.4.4}$$

From these inequalities, we immediately get that

Corollary 2.4.8 (N. Shepherd-Barron). *(1) If $p\geq 3, g\geq 3$, then $K_X^2>4(g-1)(q-1)/3$;*

(2) If $p=2, g\geq 3$, then $K_X^2>(q-1)(q-1)$.

We now begin to improve this estimation of $\lambda=K_X^2/(q-1)$ by considering its canonical system $|K_X|$.

Lemma 2.4.9. *We have $p_g>2(q-1)/3$.*

Proof. We have

$$\begin{aligned} p_g-1&=\chi(\mathcal{O}_X)-1+(q(X)-1)\\ &\geq\frac{K_X^2-4(q-1)}{12}-1+(q-1)\\ &=\frac{K_X^2+8(q-1)-12}{12}, \end{aligned} \tag{2.4.5}$$

hence $p_g>2(q-1)/3$. □

Lemma 2.4.10. *If $|K_X|$ is composed with a pencil, then $|K_X|=Z+f^*|M|$, where M is a divisor on C such that $h^0(C,M)=p_g$, and*

$$K_X^2\geq\min\{4(p_g-1)(g-1), 2(p_g+q-1)(g-1)\}.$$

Proof. Assume $|K_X|$ is composed with a pencil. If the pencil is not C, then $K_X\sim_{alg} Z+aV$, with $a\geq p_g-1$ and V is an integral divisor dominating C. So by [17] Proposition 1.3, we have either

$$K_X^2\geq 2a(p_a(V)-1)\geq 2(p_g-1)(q-1)>\lambda(X)(q-1)=K_X^2,$$

or

$$K_X^2 \geq a^2 \geq (2(q-1)/3-1)^2 > \lambda(X)(q-1) = K_X^2,$$

a contradiction. Here we have used our assumption $q-1 \gg \lambda(X)$ and Lemma 2.4.9. So the pencil is C, therefore $|K_X| = Z + f^*|M|$ and $h^0(C, M) = p_g$. The inequality

$$K_X^2 \geq K_X \cdot f^*M = (2g-2)\deg M \geq \min\{4(p_g-1)(g-1), 2(p_g+q-1)(g-1)\}$$

follows from Lemma 2.2.26. □

Theorem 2.4.11. *If $p \geq 7$, then there is a positive number ϵ (depending on p only) such that $K_X^2 \geq (p-3+\epsilon)(q-1)$.*

Proof. Since $(p-1) \mid 2g$, we have either $g \geq (p-1)$ or $g = (p-1)/2$. When $g \geq p-1$, it follows from Corollary 2.4.8 that $K_X^2 > 4(g-1)(q-1)/3 \geq 4(p-2)(q-1)/3$.

Assume $g = (p-1)/2$. If $|K_X|$ is composed with a pencil, then $K_X^2 \geq \min\{2(p_g-1)(p-3), (p_g+q-1)(p-3)\} > (p-3+\epsilon)(q-1)$ for some $\epsilon > 0$ by Lemma 2.4.9 and 2.4.10. Now we assume $|K_X|$ is not composed with pencils. Choose a general member $D' \in |K_X - Z|$. Since $D' \cdot F \leq K_X \cdot F = 2g-2 = p-3 < p$, D' is integral and separable over C by Lemma 2.2.21 and its remark. Note that Z_0 is also separable over C, we can apply Lemma 2.4.5 to $H = D' + Z_0$ to obtain

$$(K_X + H) \cdot H \geq H \cdot N.$$

Let $Z_h = \sum_i r_i E_i$, and $G = \sum_i (r_i - 1)E_i$, then $Z_0 = Z_h - G = \sum_i E_i$, so we have

$$\begin{aligned} H \cdot N =& H \cdot f^*K_C + H \cdot \Delta \\ =& 2(p-3)(q-1) - 2\sum_i (r_i-1)(q-1)\deg_K(E_i)_\eta + H \cdot \Delta \\ \geq& 2(p-3)(q-1) - 2\sum_i (r_i-1)(q-1)\deg_K(E_i)_\eta + H \cdot \Delta_h. \end{aligned}$$

On the other hand

$$
\begin{aligned}
(K_X+H)\cdot H &= 2K_X^2 - 2K_X\cdot(G+Z_v) - H\cdot(G+Z_v)\\
&\le 2K_X^2 - 2K_X\cdot G - \sum_i (r_i-1)E_i^2\\
&= 2K_X^2 - \sum_i 2(r_i-1)(p_a(E_i)-1) - K_X\cdot G\\
&\le 2K_X^2 - 2\sum_i (r_i-1)(q-1)\deg_K(E_i)_\eta - K_X\cdot G.
\end{aligned}
$$

Here we note that since E_i is separable over C, $2p_a(E_i)-2 \ge 2\deg_K(E_i)_\eta(q-1)$. Combining the two inequalities we get

$$
K_X^2 \ge (p-3)(q-1) + H\cdot\Delta_h/2 + K_X\cdot G/2. \tag{2.4.6}
$$

If $G \ne 0$, then $K_X\cdot G/(q-1)$ will be bounded from below by a positive number depending only on p (see Lemma 2.4.12 below), so by (2.4.6) $K_X^2/(q-1)-p+3$ will be bounded from below by a positive number ϵ depending on p.

Now we only have to deal with the case where $G=0$. By construction, we have $F\cdot((p-3)\Delta_h - pH)=0$, hence by Hodge Index Theorem we have

$$
((p-3)\Delta_h - pH)^2 \le 0,
$$

or

$$
(p-3)\Delta_h^2/2p + pH^2/2(p-3) \le \Delta_h\cdot H.
$$

Note that this time H is a horizontal part of an element in $|K_X|$, hence $K_X^2 \ge H^2$, so from

$$
\begin{aligned}
(3p-12)K_X^2/2(p-3) + pH^2/2(p-3) &\ge K_X^2 + H^2 \ge (K_X+H)\cdot H\\
&\ge N\cdot H \ge 2(p-3)(q-1) + H\cdot\Delta_h
\end{aligned}
$$

we see that

$$(3p-12)K_X^2/2(p-3) \geq 2(p-3)(q-1) + (p-3)\Delta_h^2/2p.$$

Combining with (2.4.3) and the fact $K_X \cdot \Delta_h + \Delta_h^2 = 2p_a(\Delta_h) - 2 \geq 2(q-1)$, we have

$$(\frac{3(p-3)}{2p} + \frac{3p-12}{2(p-3)})K_X^2 \geq (\frac{p-3}{p} + 2(p-3) + \frac{(p-3)^2}{p})(q-1),$$

or

$$K_X^2 \geq \frac{6p^2-22p+12}{6p^2-30p+27}(p-3)(q-1) > (p-3+\epsilon)(q-1).$$

□

Lemma 2.4.12. *Let B be an horizontal prime divisor with $r := [K(B) \cap K^{\mathrm{sep}} : K]$, then $K_X \cdot B + B^2 \geq 2r(q-1)$. In particular*

$$K_X \cdot B \geq \sqrt{2r(q-1)K_X^2 + (K_X^2)^2/4} - K_X^2/2 = (\sqrt{\lambda^2 + 8r\lambda} - \lambda)(q-1)/2,$$

here $\lambda = \lambda(X)$.

Proof. It is well known that $2(p_a(B)-1) \geq 2(p_a(B')-1) \geq 2r(p_a(C)-1)$, where B' is the normalisation of B (see [36], pp 289-291). So $K_X \cdot B + B^2 = 2(p_a(B)-1) \geq 2r(q-1)$.

(i) If $B^2 \leq 0$, clearly $K_X \cdot B \geq 2r(q-1) > \sqrt{2r(q-1)K_X^2 + (K_X^2)^2/4} - K_X^2/2$;

(ii) If $B^2 > 0$, B is nef and $(K_X \cdot B)^2 \geq B^2 K_X^2$, hence

$$(K_X \cdot B)^2/K_X^2 + (K_X \cdot B)^2 \geq 2r(q-1),$$

so $K_X \cdot B > \sqrt{2r(q-1)K_X^2 + (K_X^2)^2/4} - K_X^2/2$. □

Corollary 2.4.13. *If $p \geq 7$, then $\kappa_p > (p-7)/12(p-3)$.*

Next we apply this method to the cases $p = 3, 5$.

2.4.1 Case $p = 5$

When $p = 5$, we aim to work out κ_5 explicitly. The main reason why we can do this is that the smallest possible value of $g = (p-1)/2 = 2$, in which case X_η will automatically be hyperelliptic. We carry out a calculation of $\chi(\mathcal{O}_X)$ in the hyperelliptic case in the next section, which provides a more precise lower bound of $\chi(\mathcal{O}_X)/(q-1)$, and consequently gives the precise value of κ_p when $g = 2$. In this subsection, as a preparation of the proof of $\kappa_5 = 1/32$, we deal with the cases where $g > 2$ and show that $\chi/c_1^2 \geq 1/32$ also holds in these situations. Hence this result combining with the result in the next section (Theorem 2.5.7) will then imply $\kappa_5 = 1/32$ (Corollary 2.5.9).

Notice that following Noether's formula and (2.4.2), in order to prove $\chi/c_1^2 \geq 1/32$ it suffices to show $K_X^2 \geq 32(q-1)/5$. When $g \geq 6$, this inequality follows immediately from Corollary 2.4.3. So we are left to deal with the case $g = 4$ only. So we assume $g = 4$ in the rest of this subsection.

Let

$$i : H^0(X, K_X) \hookrightarrow H^0(X_\eta, K_X|_{X_\eta}) \simeq H^0(X_\eta, \omega_{X\eta/K})$$

be the canonical restriction map, $V := |K_X|$, and V_K be its restriction *i.e.* $V_K \subset |\omega_{X_\eta/K}|$ is the sub-linear system associated to the K-subspace spanned by $\mathrm{Im}(i)$ (see Subsection 2.2.3).

Lemma 2.4.14. *If $|K_X|$ is not composed with a pencil, and $D' \in |K_X - Z|$ is a general member, then either*

(1) D' is integral and separable over C; or

(2) Z_h is a section of f, and $D'^2 \geq 5(p_g - 2)$.

Proof. We consider the morphism $\phi : X_\eta \to \mathrm{P}_K^{r-1}$ defined by V_K, here r is the dimension of the K-subspace spanned by $\mathrm{Im}(i)$ (note that $r = 1$ will imply $|K_X|$ is composed with pencils). Note that by construction, we have a formula

$$\deg\phi\deg(\phi(X_\eta)) + \deg_K Z_\eta = \deg_K \omega_{X_\eta/K} = 6.$$

(1) If ϕ is separable, then D' is integral and separable over C by Lemma 2.2.21 and its remark.

(2) If ϕ is not separable, then $\deg \phi = 5$, Z_η is therefore a rational point, hence Z_h must be a section. On the other side since we have $\deg(\phi) | \deg(\phi_{|K_X - Z|})$, here $\phi_{|K_X-Z|}$ is the canonical map of X, then $\deg(\phi_{|K_X-Z|}) \geq 5$ and hence $D'^2 \geq 5(p_g - 2)$ by [17], Proposition 1.3(ii). $\square$

Theorem 2.4.15. *Under the hypothesis $g = 4$, $K_X^2 \geq 32(q-1)/5$.*

Proof. (1) If $|K_X|$ is composed with a pencil, then $|K_X| = Z + f^*|M|$, and $\deg M = p_g + q - 1 > 2(q-1)$ by Lemma 2.4.10 (Note that $\chi(\mathcal{O}_X) > 1$ and hence $p_g > q$ by Lemma 2.4.3 and assumption $q \gg 0$). So we have

$$K_X^2 \geq K_X \cdot f^*M = 6 \deg M = 6(p_g + q - 1) > 12(q-1).$$

(2) Suppose $|K_X|$ is not composed with a pencil and a general member $D' \in |K_X - Z|$ is integral and separable over C. Then $D' + Z_0$ is the sum of reduced divisors separable over C. We can apply Lemma 2.4.5 to the divisor $H := D' + Z_0$. Assume $Z_h = \sum_i r_i Z_i$, and let $G := Z_h - Z_0 = \sum_i (r_i - 1) Z_i$, then in the similar way for inequality (2.4.6), we can obtain

$$2K_X^2 \geq 12(q-1) + \sum_i (r_i - 1) K_X \cdot Z_i + H \cdot \Delta_h.$$

Note that $H \cdot \Delta_h \geq 0$ as no component of Z_0 could be inseparable over C. In particular $K_X^2/(q-1) \geq 6$ and consequently

$$K_X \cdot Z_i > (\sqrt{21} - 3)(q-1) > 3(q-1)/2$$

by Lemma 2.4.12. So if $K_X^2 \leq 32(q-1)/5$, we must have $r_i = 1$ for all i, namely $G = 0$. Then a similar trick as we did to deal with the case $G = 0$ in the proof of Theorem 2.4.11 will implies $K_X^2 > 32(q-1)/5$, contradiction.

(3) Suppose $|K_X|$ is not composed with a pencil, Z_h is a section and $D'^2 \geq$

$5(p_g - 2)$. Then

$$K_X^2 \geq D'^2 + K_X \cdot Z_h \geq 5(p_g - 2) + K_X \cdot Z_h. \tag{2.4.7}$$

In particular,

$$K_X^2 \geq 5(p_g - 2) \geq 5(K_X^2 + 8(q-1) - 24)/12,$$

hence

$$K_X^2 \geq (40(q-1) - 120)/7 \geq 39(q-1)/7,$$

as $q \gg 0$ by assumption. Combining this with Lemma 2.4.12 we obtain

$$K_X \cdot Z_h \geq (\sqrt{3705} - 39)(q-1)/14 \geq 3(q-1)/2.$$

Returning back to (2.4.7) and using (2.4.5) again, we have

$$K_X^2 \geq 5(K_X^2 + 8(q-1) - 24)/12 + 3(q-1)/2,$$

which implies $K_X^2 > 32(q-1)/5$ as $q \gg 0$ by assumption.

Lemma 2.4.14 shows that the three cases above are exhaustive. □

Corollary 2.4.16. *If $g \geq 4$, then $\chi/c_1^2 \geq 1/32$.*

2.4.2 Case $p = 3$

As another application of our method, we show $\kappa_3 > 0$ in this subsection. It suffices to prove that there is some positive number ϵ_0 independent on X such that $K_X^2 \geq (4 + \epsilon_0)(q-1)$ holds. Following Corollary 2.4.8, this inequality holds automatically if $g \geq 4$. So we divide our discussions into cases $g = 2$ and 3.

Case $g = 3$

Lemma 2.4.17. *One of the following properties is true:*

(1) $|K_X|$ *is composed with a pencil.*

(2) $|K_X|$ *is not composed with a pencil,* Z_h *is reduced and a general member* $D' \in |K_X - Z|$ *is integral and separable over* C*;*

(3) Z_h *is a section and* $(K_X - Z)^2 \geq 3(p_g - 2)$*.*

Proof. Assume $|K_X|$ is not composed with a pencil. Let $V = |K_X|$ and V_K be its restriction to the generic fibre. Then $B := Z_\eta$ is the fixed part of V_K. Let $\phi : X_\eta \to \mathbb{P}_K^{r-1}$. Note that as in case $p = 5$, we have a formula

$$\deg \phi \deg(\phi(X_\eta)) + \deg_K B = \deg_K \omega_{X_\eta/K} = 4.$$

Hence if $\deg_K B \geq 2$, we must have either $\deg \phi = 2, \deg(\phi(X_\eta)) = 1$ or $\deg \phi = 1, \deg(\phi(X_\eta)) = 2$. This first case implies that X_η is quasi-elliptic, contradiction to Lemma 2.2.19, the second implies $\phi(X_\eta)$ is a plane conic, which is indeed s-mooth since it is geometrically integral, and X_η is birational to this plane conic, contradiction. So $\deg_K B \leq 1$, hence Z_h is reduced.

If ϕ is separable, then a general member $D' \in |K_X - Z|$ is as stated in part (2) of our lemma by Lemma 2.2.21.

If ϕ is inseparable, then $\deg \phi = 3$, $\deg B = 1$. So Z_h is a section. Note that in this case the canonical map $\phi_{|K_X|} = \phi_{|K_X - Z|}$ of X is also inseparable, hence its degree is at least 3, therefore $(K_X - Z)^2 \geq 3(p_g - 2)$ by [17] Proposition 1.3. □

Theorem 2.4.18. *There is some positive constant* ϵ_0 *independent on* X *such that* $K_X^2 > (4 + \epsilon_0)(q - 1)$*.*

Proof. There are only three possibilities as below by the previous lemma.

(1). The canonical system $|K_X|$ is composed with a pencil. Then it follows from Lemma 2.4.10

$$K_X^2 \geq 4 \min\{2p_g - 2, p_g + q - 1\}.$$

Combing this inequality with (2.4.5), we have either

(1) $K_X^2 \geq 2(K_X^2 + 8(q - 1) - 12)/3$; or

(2) $K_X^2 \geq (K_X^2 + 20(q - 1))3$.

Both conditions imply that $K_X^2 \geq (4+\epsilon_0)(q-1)$ for some constant $\epsilon_0 > 0$ independent on X as $q \gg 0$.

(2). The canonical system $|K_X|$ is not composed with a pencil, Z_h is reduced and a general member $D' \in |K_X - Z|$ is integral and separable over C. So $D = D' + Z \in |K_X|$ and $D' + Z_h = D_h$. We can then apply Lemma 2.4.5 to $H = D_h$, hence

$$2K_X^2 \geq (K_X + D_h)\cdot D_h \geq N \cdot D_h \geq 8(q-1) + D_h \cdot \Pi,$$

here Π is any prime component of Δ_h. A similar trick as we did to deal with the case $G = 0$ in the proof of Theorem 2.4.11 now gives $K_X^2 \geq (4+\epsilon_0)(q-1)$ for some constant $\epsilon_0 > 0$ independent on X.

(3). The canonical system $|K_X|$ is not composed with a pencil, Z_h is a section and $(K_X - Z)^2 \geq 3(p_g - 2)$. Then we have

$$K_X^2 \geq (K_X - Z)^2 + K_X \cdot Z_h \geq 3(p_g - 2) + K_X \cdot Z_h.$$

Note that (2.4.3) implies $K_X^2 \geq 8(q-1)/3$ and hence Lemma 2.4.12 implies $K_X \cdot Z_h > 4(q-1)/3$, so we get

$$K_X^2 \geq 3(p_g - 2) + 4(q-1)/3.$$

After combining with (2.4.5) and an easy computation, this inequality will soon imply $K_X^2 \geq (4+\epsilon_0)(q-1)$ for some constant $\epsilon_0 > 0$ independent on X. □

Case $g = 2$

Lemma 2.4.19. *If $g = 2$, then Δ_h is reduced and* $\deg_K(\Delta_\eta) = 6$.

Proof. The canonical morphism of X_η/K here is automatically a flat double cover of $\mathbb{P}(H^0(X_\eta, \omega_{X_\eta/K}))$. Let $B \subseteq \mathbb{P}(H^0(X_\eta, \omega_{X_\eta/K}))$ be the branch divisor associated to this double cover, then $\deg B = 6$ by Corollary 2.2.12. Note that X_η/K is geometrically rational, so $\deg_{\widehat{K}}(B_{\widehat{K}})_{\mathrm{red}} = 2$. Hence B is either an inseparable point of degree 6, or the sum of two inseparable points of degree 3. Now since Δ_η dominates B and has the same degree over K, Δ_η must be reduced. □

Lemma 2.4.20. *The bi-canonical system $|2K_X|$ is base point free and a general member of $|2K_X|$ is integral and separable over C.*

Proof. First by [47], Theorem 25 and our assumption $K_X^2 \gg 0$, we see that $|2K_X|$ is free of base points. Everything then follows from Lemma 2.2.21 and its remark □

From this lemma, we shall apply Lemma 2.4.5 to $H = 2K_X$, hence

$$3K_X^2 \geq 4(q-1) + K_X \cdot \Delta_h. \tag{2.4.8}$$

Lemma 2.4.21. *Either*

(1) $|K_X|$ is composed with a pencil; or

(2) $|K_X|$ is not composed with a pencil, Z is vertical, and a general member $D \in |K_X - Z|$ is an integral horizontal divisor such that $D^2 \geq 2(p_g - 2)$.

Proof. Suppose $|K_X|$ is not composed with a pencil. Let $V := |K_X - Z|$, since V has horizontal part so $1 < F \cdot (K_X - Z) \leq F \cdot K_X = 2$, hence Z is vertical. It then follows from Lemma 2.2.21 and its remark that a general member $D \in V$ is integral and separable over C. Finally [17] Proposition 1.3 show that $D^2 \geq 2(p_g - 2)$ as the canonical map has degree at least 2 in this case. □

Theorem 2.4.22. *We have $K_X^2 > (4 + \epsilon_0)(q-1)$ for some positive constant ϵ_0 independent on X.*

Proof. (1) If $|K_X|$ is composed with a pencil, then $|K_X| = Z + f^*|M|$, and $\deg M \geq \min\{2p_g - 2, p_g + q - 1\}$ (Lemma 2.4.10). Note in this case that the components Δ_h is different from any component of Z for sake of degree over C, so

$$K_X \cdot \Delta \geq K_X \cdot \Delta_h = Z \cdot \Delta_h + 6 \deg M \geq 6 \deg M.$$

Hence (2.4.8) shows that

$$3K_X^2 \geq 4(q-1) + 6 \deg M.$$

After combining this with (2.4.5) and an easy computation we obtain

$$K_X^2 \geq (4+\epsilon_0)(q-1).$$

(2) Suppose $|K_X|$ is not composed with a pencil. Let $D \in |K_X - Z|$ be a general member. By Lemma 2.4.5, we have

$$(K_X + D)\cdot D \geq N \cdot D \geq 4(q-1) + D\cdot\Delta. \tag{2.4.9}$$

Since by construction $(3D - \Delta_h)\cdot F = 0$, we have $(3D-\Delta_h)^2 \leq 0$, *i.e.*

$$D\cdot\Delta_h \geq 3D^2/2 + \Delta_h^2/6.$$

Combining this with(2.4.9) and Lemma 2.4.21 we see that

$$\begin{aligned} K_X^2 &\geq (K_X+D)\cdot D - D^2 \geq 4(q-1) + D\cdot\Delta_h - D^2 \\ &\geq 4(q-1) + D^2/2 + \Delta_h^2/6 \geq 4(q-1) + p_g - 2 + \Delta_h^2/6. \end{aligned}$$

Combining with (2.4.8) and (2.4.5), we obtain

$$\begin{aligned} 3K_X^2/2 &= (3K_X^2)/6 + K_X^2 \\ &\geq (4+4/6)(q-1) + p_g - 2 + (\Delta_h^2 + K_X\cdot\Delta_h)/6 \\ &\geq 16(q-1)/3 + p_g - 2 \\ &\geq 16(q-1)/3 + (K_X^2 + 8(q-1))/12 - 2. \end{aligned}$$

Hence $K_X^2 \geq 72(q-1)/17 - 24/17 \geq (4+\epsilon_0)(q-1)$ as $q \gg 0$. □

Corollary 2.4.23. *We have $\kappa_3 > 0$.*

To close this section, we mention that if combine all the theorems proven in this section, we get a proof of Theorem 2.1.3 except for the last statement $\kappa_5 = 1/32$.

2.5 Case of hyperelliptic Fibration

We keep the notations of Section 4 a)-h). In this section, we calculate $\chi(\mathcal{O}_X)$ directly under the assumption $p \geq 5, g = (p-1)/2$ and X_η is quasi-hyperelliptic. Our calculation will show that our conjectural κ_p is indeed the best bound of χ/c_1^2 for these surfaces. It is natural to believe that those surfaces whose $\chi/(c_1^2)$ approaches κ_p should appear in the case $g = (p-1)/2$, the smallest possible value of g, so somehow we have proven our conjecture for the "hyperelliptic part".

From now on we assume X_η is quasi-hyperelliptic and $g = (p-1)/2$.

By our assumption X_η is a flat double cover of a smooth plane conic P. Let $B \subset P$ be the branch divisor of this flat double cover, then $\deg B = p+1$ by Corollary 2.2.12. Since X_η/K is normal but not geometrically normal by assumption, B/K is reduced but not geometrically reduced (Proposition 2.2.13). Therefore B contains at least one inseparable point. Consequently B is the sum of a rational point and an inseparable point of degree p, in particular $P \simeq \mathbb{P}^1_K$.

We then identify P with the generic fibre of $p_1 : Z = \mathbb{P}^1_C \to C$ in a way such that the rational point contained in B is the infinity point. Here we denote by U, V the two homogeneous coordinates of $\mathbb{P}^1$, and ∞ is defined by $V = 0$. Denote by $\varTheta_K$ the inseparable point contained in B, so Θ_K is defined by $U^p - hV^p$ for a certain element $h \in K \backslash K^p$.

Let X_0 be the normalisation of Z in $K(X)$, and let Π be the branch divisor associated to this flat double cover $X_0 \to Z$, then $B = \Pi|_{\mathbb{P}^1_K}$. Define Π_1 (resp. Π_2) to be the closure of $\infty \in B$ (resp. $\varTheta_K \in B$) in Z and Π_3 to be the remaining vertical branch divisors.

Here by abuse of language we denote by h not only the element of K mentioned above to define $\varTheta_K$ but also the unique morphism $h : C \to \mathbb{P}^1_k$ that maps $u = U/V$ to h in function fields. Define $\alpha := \deg(h)$ and $A := h^*(\infty)$, it is clear that $\deg A = \alpha$.

With some local computations we immediately obtain the next proposition on the configuration of Π.

Proposition 2.5.1. *We have*

(1) $\Pi_1 = C \times_k \infty$, *and* $\Pi_1 \cap \Pi_2$ *equals to* $A \times \infty$.

(2) $\mathcal{O}_Z(\Pi_2) = \mathcal{O}(p) \otimes \mathcal{O}_Z(p_1^* A)$, *the canonical morphism* $\Pi_2 \to C$ *is a homeomorphism, and the singularities of* Π_2 *are exactly the pre-image of points on* C *where the morphism* h *is ramified.*

(3) $\mathcal{O}_Z(\Pi_1) = \mathcal{O}(1)$, $\Pi_3 = p_1^* D$, *for a reduced divisor* D. *Let* $d := \deg D$, *then* $\alpha + d$ *is even, and* $\mathcal{O}_Z(\Pi) = \mathcal{O}(p+1) \otimes p_1^* \mathcal{O}_C(A + D)$.

(4) $\chi(\mathcal{O}_{X_0}) = (p-3)(q-1)/2 + (p-1)(\alpha+d)/4$.

Here we note that the last statement comes from Corollary 2.2.2.

We are going to run the canonical resolution of singularities (Definition 2.2.15) to $X_0 \to Z$ to obtain $\chi(\mathcal{O}_X)$. We first need to analyze the singularities of Π. From the above proposition, non-negligible singularities of Π are all lying on Π_2. Since Π_2 is homeomorphic to C via p_1, we shall use following conventions: if $b_2 \in \Pi_2$ is a singularity of Π, we divide it into one of the 4 types below according to its image $b := p_1(b_2) \in C$, and use the notation ξ_b to denote ξ_{b_2} (see Definition 2.2.15 and Definition 2.2.16, here the flat double cover is taken to be $X_0 \to Z$). The 4 types of singularities are:

Type I : $b \notin (A \cup D)$ and b is a ramification of h. The local function of Π near b_2 is $u^p - h$ in $\mathcal{O}_{b,C}[u]$.

Type II : $b \in D \backslash A$. The local function of Π near b_2 is $t(u^p - h)$ in $\mathcal{O}_{b,C}[u]$, where t is a uniformizer of $\mathcal{O}_{b,C}$.

Type III : $b \in A \backslash D$. The local function of Π near b_2 is $v(v^p - 1/h)$ in $\mathcal{O}_{b,C}[v]$

Type IV : $b \in (A \cap D)$. The local function of Π near b_2 is $tv(v^p - 1/h)$ in $\mathcal{O}_{b,C}[v]$, where t is a uniformizer of $\mathcal{O}_{b,C}$.

Denote

$\mathcal{S} := \{b \mid b$ is of type I, II, III or IV$\}$;

$\mathcal{T} := \{b \mid b$ is of type III or IV, and h is unramified or tamely ramified at $b\}$;

$\mathcal{W} := \{b \mid b$ is of type III or IV, and h is wildly ramified at $b\}$.

By (2.2.10),

$$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{X_0}) - \sum_{b \in \mathcal{S}} \xi_b = \frac{(p-3)(q-1)}{2} + \frac{(p-1)(\alpha+d)}{4} - \sum_{b \in \mathcal{S}} \xi_b.$$

Set

$$d_b := \begin{cases} 1, & b \in D; \\ 0, & b \notin D. \end{cases}$$

Then

$$\chi(\mathcal{O}_X) = \frac{(p-3)(q-1)}{2} + \frac{(p-1)\alpha}{4} + \sum_{b \in \mathcal{S}} (\frac{(p-1)d_b}{4} - \xi_b). \tag{2.5.1}$$

Next we study in detail these four kinds of singularities. We will find a relation between $\frac{(p-1)d_b}{4} - \xi_b$ and $R_b(h)$ for all b. In order to do this, we give a definition as follows.

Definition 2.5.2 Suppose $b \in C$ is a closed point, $t \in \mathcal{O}_{b,C}$ is a uniformizer, v is the canonical discrete valuation and $e \in t\mathcal{O}_{b,C} \backslash \mathcal{O}_{b,C}^p$, we consider an arbitrary flat double cover $S_0 \to Y_0 := \mathrm{Spec}(\mathcal{O}_{b,C}[x])$ with branch divisor $B_0 = \mathrm{div}(x^p - e)$ (resp. $\mathrm{div}(t(x^p - e))$, $\mathrm{div}(x(x^p - e))$, $\mathrm{div}(tx(x^p - e))$). Let Q denote the point (x, t) of $\mathrm{Spec}(\mathcal{O}_{b,C})[x]$, then we define the number $\xi_{I,e}$ (resp. $\xi_{II,e}$, $\xi_{III,e}$, $\xi_{IV,e}$) to be ξ_Q with respect to this flat double (see Definition 2.2.16).

Note that by definition we have that $\xi_b = \xi_{*,e}$ for some e such that $\Lambda_b(e) = \Lambda_b(h)$ (see Definition 2.2.23), here $*$ is the type of b (*i.e.* I, II, III or IV).

Note also that if $R_b(e) \geq p$ (see Definition 2.2.23), then $e = t^p(\lambda_1 + e_1)$ for a unique $\lambda_1 \in k$ and $e_1 \in t\mathcal{O}_{b,C}$. In particular $R_b(e_1) = R_b(e) - p$, and $\lambda_1 \neq 0$ if and only if $v(e) = p$. If we blow up $y_0 = Q$ to obtain the first step of the canonical resolution (see Definition 2.2.15), we can obtain a recursion relation as follows.

Lemma 2.5.3. *(1) If $R_b(e) \geq p$, then*

$$\xi_{I,e} = \frac{(p-1)(p-3)}{8} + \xi_{II,e_1}, \ \xi_{II,e} = \frac{(p-1)(p+1)}{8} + \xi_{I,e_1}; \tag{2.5.2}$$

(2) If $R_b(e) \geq p$ and $v(e) > p$, then

$$\xi_{III,e} = \frac{(p-1)(p+1)}{8} + \xi_{III,e_1}, \ \xi_{IV,e} = \frac{(p-1)(p+1)}{8} + \xi_{IV,e_1}. \tag{2.5.3}$$

(3) If $R_b(e) \geq p$ and $v(e) = p$, then

$$\xi_{III,e} = \frac{(p-1)(p+1)}{8} + \xi_{I,e_1},\ \xi_{IV,e} = \frac{(p-1)(p+1)}{8} + \xi_{II,e_1}. \tag{2.5.4}$$

Proof. According to the process of canonical resolution, after blowing-up we can get the two tables (TABLE 1 & 2) below, everything then follows from the table. We remark that it is clear outside the open subset $\mathrm{Spec}(\mathcal{O}_{b,C}[x/t])$, B_1 could have at worst negligible singularities. □

	m_0	l_0	equation of B_1 on $\mathrm{Spec}(\mathcal{O}_{b,C}[x/t])$	equation of singularities	$(l_0^2 - l_0)/2$
I	p	$\frac{p-1}{2}$	$t((x/t)^p - e_1)$	$t((x/t)^p - e_1)$	$\frac{(p-1)(p-3)}{8}$
II	$p+1$	$\frac{p+1}{2}$	$(x/t)^p - e_1$	$(x/t)^p - e_1$	$\frac{(p-1)(p+1)}{8}$
III	$p+1$	$\frac{p+1}{2}$	$(x/t)((x/t)^p - e_1)$	$(x/t)((x/t)^p - e_1)$	$\frac{(p-1)(p+1)}{8}$
IV	$p+2$	$\frac{p+1}{2}$	$t(x/t)((x/t)^p - e_1)$	$t(x/t)((x/t)^p - e_1)$	$\frac{(p-1)(p+1)}{8}$

表 2.1: table of $\lambda_1 = 0$.

	m_0	l_0	equation of B_1 on $\mathrm{Spec}(\mathcal{O}_{b,C}[x/t])$	equation of singularities	$(l_0^2 - l_0)/2$
I	p	$\frac{p-1}{2}$	$t((x/t - \lambda_1^{1/p})^p - e_1)$	$t((x/t - \lambda_1^{1/p})^p - e_1)$	$\frac{(p-1)(p-3)}{8}$
II	$p+1$	$\frac{p+1}{2}$	$(x/t - \lambda_1^{1/p})^p - e_1$	$(x/t - \lambda_1^{1/p})^p - e_1$	$\frac{(p-1)(p+1)}{8}$
III	$p+1$	$\frac{p+1}{2}$	$(x/t)((x/t - \lambda_1^{1/p})^p - e_1)$	$(x/t - \lambda_1^{1/p})^p - e_1$	$\frac{(p-1)(p+1)}{8}$
IV	$p+2$	$\frac{p+1}{2}$	$t(x/t)((x/t - \lambda_1^{1/p})^p - e_1)$	$t((x/t - \lambda_1^{1/p})^p - e_1)$	$\frac{(p-1)(p+1)}{8}$

表 2.2: table of $\lambda_1 \neq 0$.

Lemma 2.5.4. *(1) The number $\xi_{*,e}$ depends on the ramification type $\Lambda_b(e)$ (see Definition 2.2.23) rather than e itself.*

(2) If $*$ is I or II, then $\xi_{*,e}$ depends on $R_b(e)$ only.

Since $\xi_{*,e}$ depends on the ramification type $\Lambda_b(e)$ rather than e, we shall also write $\xi_{*,\Lambda}$ to denote $\xi_{*,e}$ for those e with $\Lambda_b(e) = \Lambda$.

Proof. By Lemma 2.5.3 if $R_b(e) \geq p$ then $\xi_{*,e}$ is determined by ξ_{*,e_1} and whether $\lambda_1 = 0$ or not. However it is clear that the ramification type of $\Lambda_b(e)$ is also determined by $\Lambda_b(e_1)$ and whether $\lambda_1 = 0$ or not, so the our lemma is true if it is true for cases $R_b(e) < p$, the latter is clear. $\square$

Lemma 2.5.5. *(1) For any e, we have*

$$\frac{(p-1)^2 R_b(e)}{8p} - \xi_{I,e} \geq 0; \tag{2.5.5}$$

$$\frac{(p-1)^2 R_b(e)}{8p} + \frac{p-1}{4} - \xi_{II,e} \geq 0; \tag{2.5.6}$$

(2) For any e with tame ramification, we have

$$\frac{(p-1)(p+1) R_b(e)}{8p} - \xi_{III,e} \geq -\frac{p-1}{4p}; \tag{2.5.7}$$

$$\frac{(p-1)(p+1) R_b(e)}{8p} + \frac{p-1}{4} - \xi_{IV,e} \geq -\frac{p-1}{4p}; \tag{2.5.8}$$

(3) If e has wild ramification and $\Lambda_b(e) = \{pj, R_b(e)\}$, then

$$\frac{(p-1)^2 R_b(e)}{8p} - \xi_{III,e} \geq -\frac{(p-1)j}{4}; \tag{2.5.9}$$

$$\frac{(p-1)^2 R_b(e)}{8p} + \frac{p-1}{4} - \xi_{IV,e} \geq -\frac{(p-1)j}{4}. \tag{2.5.10}$$

We shall put the proof to the next section as it is a bit long.

Lemma 2.5.6. $\alpha = \sum_{b\in\mathcal{T}} (R_b(h) + 1) + \sum_{b\in\mathcal{W}} (pj_b(h))$.

Proof. By definition

$$A = \sum_{b\in\mathcal{T}} (R_b(h) + 1)b + \sum_{b\in\mathcal{W}} pj_b(h)b.$$

Taking degree we obtain our lemma. $\square$

Theorem 2.5.7. *Under the assumption $g = (p-1)/2$ and X_η being quasi-hyperelliptic, we have $\chi(\mathcal{O}_X) \geq (p^2 - 4p - 1)(q - 1)/4p$.*

Proof. By Equation (2.5.1)

$$\chi(\mathcal{O}_X) = \frac{(p-3)(q-1)}{2} + \frac{(p-1)(\alpha+d)}{4} - \sum_{b \in \mathcal{S}} \xi_b.$$

Lemma 2.5.5 and Lemma 2.5.6 show that

$$\begin{aligned}\frac{(p-1)d}{4} - \sum_{b\in\mathcal{S}} \xi_b &\geq -\sum_{b\in\mathcal{S}} \frac{(p-1)^2 R_b(h)}{8p} - \sum_{b\in\mathcal{T}} \frac{(p-1)(R_b(h)+1)}{4p} - \sum_{b\in\mathcal{W}} \frac{(p-1)j}{4} \\ &= -\sum_{b\in\mathcal{S}} \frac{(p-1)^2 R_b(h)}{8p} - \frac{(p-1)\alpha}{4p}.\end{aligned}$$

Hence

$$\begin{aligned}\chi(\mathcal{O}_X) &= \frac{(p-3)(q-1)}{2} + \frac{(p-1)(\alpha+d)}{4} - \sum_{b\in\mathcal{S}} \xi_b \\ &\geq \frac{(p-3)(q-1)}{2} + \frac{(p-1)\alpha}{4} - \sum_{b\in\mathcal{S}} \frac{(p-1)^2 R_b(h)}{8p} - \frac{(p-1)\alpha}{4p} \\ &= \frac{(p^2-4p-1)(q-1)}{4p},\end{aligned}$$

by Hurwitz's formula:

$$2\alpha + 2(q-1) = \sum_{b\in\mathcal{S}} R_b(h). \tag{2.5.11}$$

□

Corollary 2.5.8. *Under the assumption $g = (p-1)/2$ and X_η being quasi-hyperelliptic, the optimal bound of χ/c_1^2 is $(p^2 - 4p - 1)/4(3p^2 - 8p - 3)$.*

Proof. Since $\chi(\mathcal{O}_X) \geq (p^2 - 4p - 1)(q - 1)/4p$, we see that

$$\frac{\chi(\mathcal{O}_X)}{K_X^2} = \frac{\chi(\mathcal{O}_X)}{12\chi(\mathcal{O}_X) - c_2(X)} \geq \frac{\chi(\mathcal{O}_X)}{12\chi(\mathcal{O}_X) + 4(q-1)} \geq \frac{p^2-4p-1}{4(3p^2-8p-3)}.$$

On the other hand, Raynaud's example in Subsection 2.3.1 gives examples whose

χ/c_1^2 is equal to $(p^2-4p-1)/4(3p^2-8p-3)$. $\square$

Corollary 2.5.9. *We have* $\kappa_5 = 1/32$.

Proof. When $g=(p-1)/2$, X_η is automatically hyperelliptic, hence the best bound of χ/c_1^2 is $1/32$ for these surfaces. Combining this with Corollary 2.4.16, we obtain $\kappa_5 = 1/32$. $\square$

2.6 Proof of Lemma 2.5.5

Assume $\mathrm{char}(k) \neq 2$, $a, b \in \{0,1\}$, and $m, n \in \mathbb{N}_+$ are two numbers co-prime to each other. Let $S := \mathrm{Spec}(k[\![x,y,t]\!]_{(x,y,t)}/(y^2 - x^a t^b (x^m - t^n)))$ and $f : \widetilde{S} \to S$ be an arbitrary desingularization, we define $\xi(a,b,m,n) := \dim_k R^1 f_* \mathcal{O}_{\widetilde{S}}$.

Proposition 2.6.1. *If* $2 \nmid m$, *then*

$$\xi(a,b,m,n) \leq \frac{(m-1)^2(n-1)}{8m} + \frac{(m-1)n}{4m}a + \frac{m-1}{4}b.$$

First we point out an algorithm of calculating of $\xi(a,b,m,n)$.

Lemma 2.6.2. *(1) If* $m=1$ *or* $n=1$, $\xi(a,b,m,n)=0$;

(2) If $m>n>1$, *then*

$$\xi(a,b,m,n) = \begin{cases} \xi(0,b,m-n,n) + (a+b+n)(a+b+n-2)/8, \\ \qquad \text{if } 2 \mid a+b+n; \\ \xi(1,b,m-n,n) + (a+b+n-1)(a+b+n-3)/8, \\ \qquad \text{if } 2 \nmid a+b+n. \end{cases}$$

(3) If $n>m>1$, *then*

$$\xi(a,b,m,n) = \begin{cases} \xi(a,0,m,n-m) + (a+b+m)(a+b+m-2)/8, \\ \qquad \text{if } 2 \mid a+b+m; \\ \xi(a,1,m,n-m) + (a+b+m-1)(a+b+m-3)/8, \\ \qquad \text{if } 2 \nmid a+b+m. \end{cases}$$

Proof. In fact S is obtained as a flat double cover of $Y := \mathrm{Spec}(k[x,t])$ with branch divisor $B = \mathrm{div}(x^a t^b (x^m - t^n))$. Our lemma follows from the process of the canonical resolution(see Definition 2.2.15). □

Lemma 2.6.3. *Proposition 2.6.1 holds if it holds for all $n < m$.*

Proof. Let $n = m + n'$. If $2 \mid a + b + m$, then by Lemma 2.6.2 we have

$$\begin{aligned}&\frac{(m-1)^2(n-1)}{8m} + \frac{(m-1)n}{4m}a + \frac{m-1}{4}b - \xi(a,b,m,n)\\ \geq &\frac{(m-1)^2(n'-1)}{8m} + \frac{(m-1)n'}{4m}a - \xi(a,0,m,n').\end{aligned}$$

If $2 \nmid a + b + m$, then we also have

$$\begin{aligned}&\frac{(m-1)^2(n-1)}{8m} + \frac{(m-1)n}{4m}a + \frac{m-1}{4}b - \xi(a,b,m,n)\\ \geq &\frac{(m-1)^2(n'-1)}{8m} + \frac{(m-1)n'}{4m}a + \frac{m-1}{4} - \xi(a,1,m,n').\end{aligned}$$

So it is sufficient to prove the inequality for pair (m, n'). □

Proof of Proposition 2.6.1. We shall proceed by induction on m. When $m = 1$, the statement holds trivially. Assume our proposition holds for odd numbers smaller than m, we need to show it also holds for m. By Lemma 2.6.3, we can assume $n < m$.

If $2 \nmid n$, then

$$\begin{aligned}\xi(a,b,m,n) = \xi(b,a,n,m) &\leq \frac{(n-1)^2(m-1)}{8n} + \frac{(n-1)m}{4n}b + \frac{n-1}{4}a\\ &\leq \frac{(m-1)^2(n-1)}{8m} + \frac{m-1}{4}b + \frac{(m-1)n}{4m}a.\end{aligned}$$

If $2 \mid n$, let $m = n + m'$, then by Lemma 2.6.2, we have

$$\begin{aligned}
\xi(a,0,m,n) &= \xi(a,0,m',n) + \frac{n(n-2)}{8} \\
&\leq \frac{(m'-1)^2(n-1)}{8m'} + \frac{(m'-1)n}{m'}a + \frac{n(n-2)}{8} \\
&< \frac{(m-1)^2(n-1)}{8m} + \frac{(m-1)n}{m}a
\end{aligned}$$

$$\begin{aligned}
\xi(0,1,m,n) &= \xi(1,1,m',n) + \frac{n(n-2)}{8} \\
&\leq \frac{(m'-1)^2(n-1)}{8m'} + \frac{(m'-1)n}{4m'} + \frac{m'-1}{4} + \frac{n(n-2)}{8} \\
&< \frac{(m-1)^2(n-1)}{8m} + \frac{m-1}{4}
\end{aligned}$$

$$\begin{aligned}
\xi(1,1,m,n) &= \xi(0,1,m',n) + \frac{n(n+2)}{8} \\
&\leq \frac{(m'-1)^2(n-1)}{8m'} + \frac{m'-1}{4} + \frac{n(n+2)}{8} \\
&\leq \frac{(m-1)^2(n-1)}{8m} + \frac{(m-1)n}{m} + \frac{m-1}{4}.
\end{aligned}$$

Here we note that

$$\frac{(m-1)^2(n-1)}{8m} - \frac{(m'-1)^2(n-1)}{8m'} = \frac{n(n-1)}{8} - \frac{n(n-1)}{8mm'},$$

and the last equality holds only if $n = m - 1$. $\square$

Proof of Lemma 2.5.5. The previous Proposition asserts our statements a) and b) immediately by Lemma 2.5.3.

For c), we assume $\Lambda(e) = \{pj, R_b(e)\}$, then $e = t^{pj}(\lambda + e')$, with $\lambda \neq 0$. Hence by Lemma 2.5.3 we have:

$$\begin{aligned}\frac{(p-1)^2R_b(e)}{8p}-\xi_{III,e} &= \frac{(p-1)^2R_b(e')}{8p}-\xi_{I,e'}+\frac{(p-1)^2j}{8}-\frac{(p+1)(p-1)j}{8}\\ &= (\frac{(p-1)^2R_b(e')}{8p}-\xi_{I,e'})-\frac{(p-1)j}{4}\\ &\geq -\frac{(p-1)j}{4},\end{aligned}$$

and similarly

$$\frac{(p-1)^2R_b(e)}{8p}+\frac{p-1}{4}-\xi_{IV,e}\geq -\frac{(p-1)j}{4}.$$

□

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